Formule di quadratura numerica

$$\int_a^b f(x)dx = \sum_{i=0}^{n} \omega_i f(x_i) + R_n$$

$\{x_i\}$ nodi

$\{\omega_i\}$ pesi

Resto della formula

Formule Interpolatorie

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \int_a^b E_n(x)dx$$

La formula è esatta se

$$R_n = \int_a^b E_n(x)dx = 0$$

GRADO DI PRECISIONE: Misura l'accuratezza di una formula di quadratura.

Tale misura viene data in relazione alla capacità della formula stessa di valutare correttamente l'integrale quando f(x) è un polinomio.

DEFINIZIONE

Una formula ha grado di precisione **p** se applicata a polinomi di grado **<=** **p** è esatta e se esiste un polinomio di grado **p+1** per cui non lo è.

$$
\begin{cases}
\int_a^b dx = \sum_{i=0}^{p} \omega_i = \omega_0 + \omega_1 + \dots + \omega_p \\
\int_a^b x\,dx = \sum_{i=0}^{p} \omega_i x_i = \omega_0 x_0 + \omega_1 x_1 + \dots + \omega_p x_p \\
\int_a^b x^2 dx = \sum_{i=0}^{p} \omega_i x_i^2 = \omega_0 x_0^2 + \omega_1 x_1^2 + \dots + \omega_p x_p^2 \\
\vdots \\
\int_a^b x^p dx = \omega_0 x_0^p + \omega_1 x_1^p + \dots + \omega_p x_p^p
\end{cases}
$$

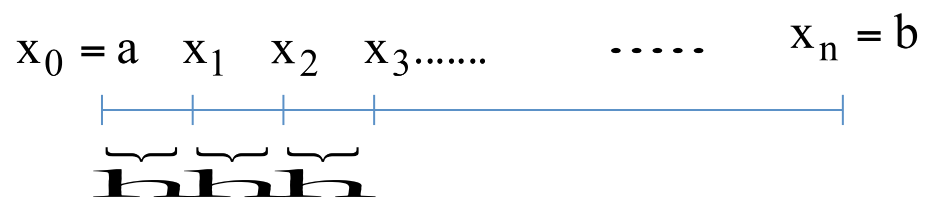
$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_p \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_p^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^p & x_1^p & x_2^p & \cdots & x_p^p \end{pmatrix}}_{\text{VANDERMONDE}} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_p \end{pmatrix} = \begin{pmatrix} \int_a^b dx \\ \int_a^b x\,dx \\ \int_a^b x^2 dx \\ \vdots \\ \int_a^b x^p dx \end{pmatrix}$$

$$x_i \neq x_j \qquad \exists! \ \{\omega_i\}_{i=0}^{p}$$

Quindi fissati p+1 nodi distinti $\exists!$
la formula di quadratura con grado di precisione almeno p

Formule di Newton-Cotes

- **Formule interpolatorie**
- **Nodi equidistanti**



$$x_i = x_0 + ih \qquad i = 0,1,\dots,n \qquad x = x_0 + th \qquad t \in R^+$$

$$x - x_1 = (x_0 + th) - (x_0 + h) = h(t-1)$$
$$x - x_2 = (x_0 + th) - (x_0 + 2h) = h(t-2)$$
$$\vdots$$
$$x - x_n = (x_0 + th) - (x_0 + nh) = h(t-n)$$

$$x_i - x_0 = (x_0 + ih) - x_0 = ih$$
$$x_i - x_1 = (x_0 + ih) - (x_0 + h) = h(i-1)$$
$$x_i - x_2 = (x_0 + ih) - (x_0 + 2h) = h(i-2)$$
$$\vdots$$
$$x_i - x_n = (x_0 + ih) - (x_0 + nh) = h(i-n)$$

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i)$$

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \int_a^b E_n(x)dx$$

$$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

$$\int_a^b f(x)dx \approx \int_a^b \sum_{i=0}^{n} l_i(x) f(x_i)dx = \sum_{i=0}^{n} \underbrace{\int_a^b l_i(x)dx}_{\omega_i}\, f(x_i)$$

$$\boxed{\omega_i = \int_a^b l_i(x)dx \qquad i = 0,1,\ldots,n}$$

Pesi della formula

$$l_i(x) = \frac{(x - x_0)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}$$

$a = x_0 \quad$ per $t = 0$

$b = x_n \quad$ per $t = n$

$x = x_0 + th$

$dx = hdt$

$$\boxed{\omega_i = \int_a^b l_i(x)dx = h\int_0^n \frac{(ht)h(t-1)h(t-2)\dots h(t-n)}{(hi)h(i-1)h(i-2)\dots h(i-n)}dt = h\int_0^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(t-j)}{(i-j)}dt}$$

Pesi delle formule
di Newton-Cotes

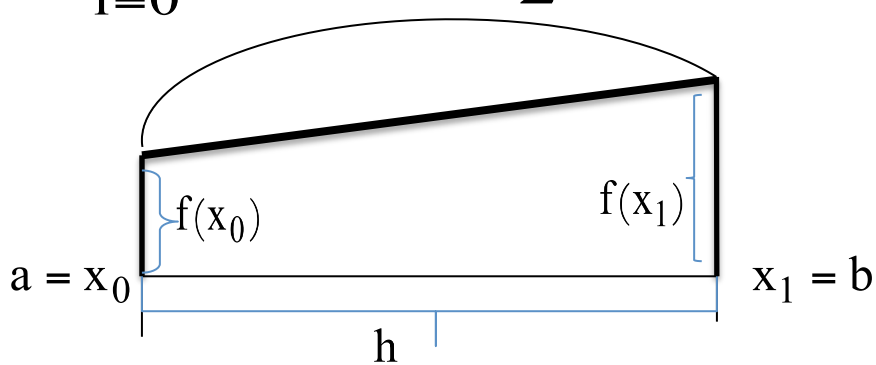
Formule di Newton-Cotes a due punti

$$n = 1$$
$$h = b - a$$

$$\omega_0 = \int_a^b l_0(x)dx = h\int_0^1 \frac{(t-1)}{-1}dt = h\int_0^1 (1-t)dt = \frac{h}{2} = \frac{b-a}{2}$$

$$\omega_1 = \int_a^b l_1(x)dx = h\int_0^1 t dt = \frac{h}{2} = \frac{b-a}{2}$$

$$\int_a^b f(x)dx \approx \sum_{i=0}^{1} \omega_i\, f(x_i) = \frac{h}{2}[f(x_0) + f(x_1)]$$



Formula del trapezio

$$R_n = \int_a^b E_n(x)dx$$

$$E_n(x) = (x - x_0)(x - x_1)\ldots(x - x_{n-1})\ldots(x - x_n)\frac{f^{(n+1)}(\zeta)}{(n+1)!}$$

$$R_n = \int_a^b \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-x_0)(x-x_1)\ldots.(x-x_n)dx = \frac{f^{(n+1)}(\zeta)}{(n+1)!}\int_a^b (x-x_0)(x-x_1)\ldots.(x-x_n)dx$$

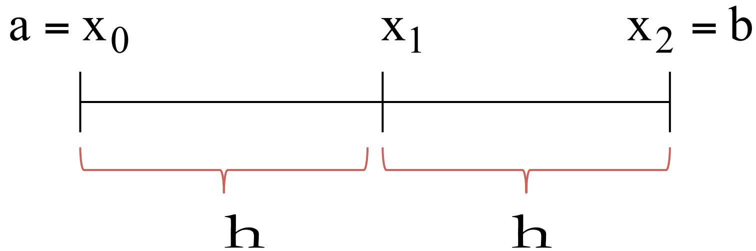
$$n = 1$$

$$\boxed{R_1 = \frac{f''(\zeta)}{2}\int_a^b (x - x_0)(x - x_1)dx = -\frac{h^3}{12}f''(\zeta)}$$

GRADO DI PRECISIONE **p=1**

Formule di Newton-Cotes a tre punti (Cavalieri-Simpson)

$$n = 2 \qquad h = \frac{b-a}{2}$$



$$\omega_0 = \int_a^b l_0(x)dx = h\int_0^2 \frac{(t-1)(t-2)}{(-1)(-2)}dt = \frac{h}{2}\int_0^2 t^2 - 2t - t + 2dt = \ldots = \frac{h}{3}$$

$$\omega_1 = \int_a^b l_1(x)dx = h\int_0^2 \frac{t(t-2)}{(1)(1-2)}dt = \ldots = \frac{4}{3}h$$

$$\omega_2 = \int_a^b l_2(x)dx = h\int_0^2 \frac{t(t-1)}{2}dt = \ldots = \frac{h}{3}$$

$$\int_a^b f(x)dx \approx \sum_{i=0}^{2} \omega_i \, f(x_i) = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$R_2 = -\frac{h^5}{90} f^{IV}(\zeta)$$

GRADO DI PRECISIONE p=3

Dal teorema generalizzato della media si ricava

n dispari

GRADO DI PRECISIONE **n**

$$\boxed{R_n = \frac{f^{(n+1)}(\zeta)}{(n+1)!} h^{n+2} \int_0^n \prod_{j=0}^{n} (t-j)\,dt}$$

n pari

GRADO DI PRECISIONE **n+1**

$$\boxed{R_n = \frac{f^{(n+2)}(\zeta)}{(n+2)!} h^{n+3} \int_0^n t \prod_{j=0}^{n} (t-j)\,dt}$$

Esempio 1

$$I = \int_0^3 x^2 dx = 9$$

TRAPEZI (G.P.= 1)

$$\tilde{I} = \frac{h}{2}[f(0) + f(3)] = \frac{3}{2}.9 = 13.5 \qquad h = 3$$

CAVALIERI-SIMPSON (G.P. =3)

$$\tilde{I} = \frac{h}{3}[f(0) + 4f(\frac{3}{2}) + f(3)] = \frac{1}{2}.18 = 9 \qquad h = \frac{3}{2}$$

Esempio 2

$$I = \int_0^1 \frac{1}{1+x} dx = 0.69314718$$

TRAPEZI (G.P.= 1) $h = 1$

$$\tilde{I} = \frac{1}{2}[f(0) + f(1)] = \frac{1}{2}.[1 + \frac{1}{2}] = \frac{3}{4} = 0.75$$

CAVALIERI-SIMPSON (G.P. =3) $h = \frac{1}{2}$

$$\tilde{I} = \frac{1}{6}[f(0) + 4f(\frac{1}{2}) + f(1)] = \frac{25}{36} = 0.6944$$

E’ possibile affermare che aumentando n l’approssimazione dell’integrale con una formula di Newton-Cotes migliori?

Ciò è vero fino ad **n=7**

Problemi dovuti ad errori di arrotondamento rendono numericamente non affidabili le formule

POSSIBILE ALTERNATIVA

↓

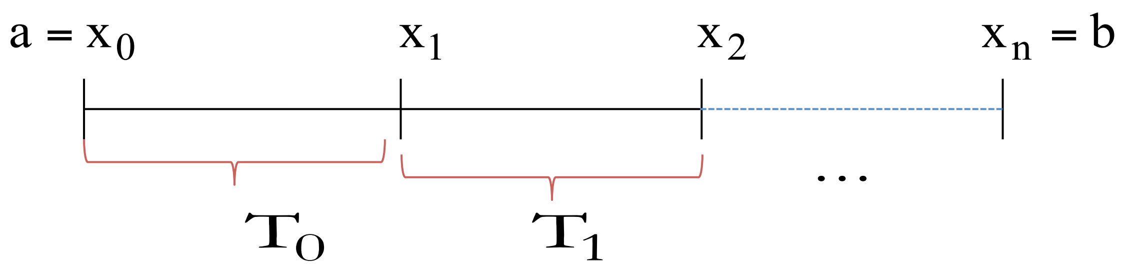
FORMULE COMPOSTE

FORMULE COMPOSTE (o GENERALIZZATE)

$$\{x_0, x_1, \dots, x_n\}$$

$$T_k = [x_k, x_{k+1}] \qquad k = 0,1,\dots,n-1$$

FORMULE COMPOSTE (o GENERALIZZATE)



$$\tilde{I} = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \dots + \frac{h}{2}[f(x_{n-1}) + f(x_n)] =$$

$$\tilde{I} = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)]$$

$$\boxed{R = \sum_{i=0}^{n-1} -\frac{(b-a)^3}{12n^3} f''(\zeta_i) \approx -\frac{(b-a)^3}{12n^2} f''(\zeta)}$$

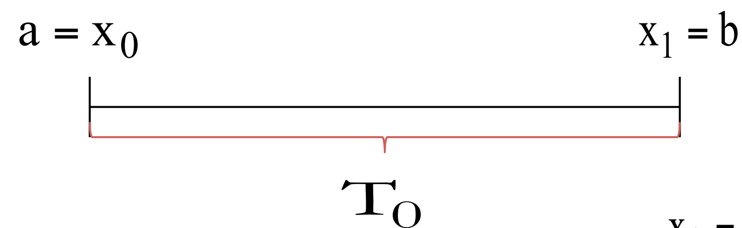
Esempio 3

TRAPEZI COMPOSTO

$$I=\int_0^3 x^2 dx = 9$$

$h = 3$

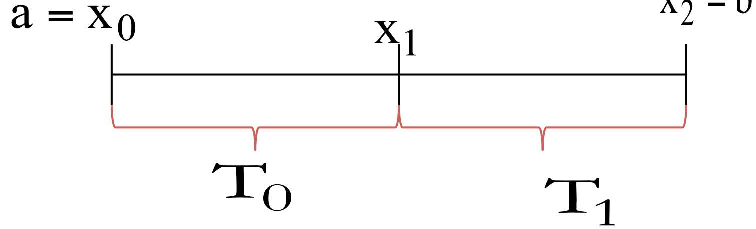
$n = 1$



$$\tilde{I} = \frac{h}{2}[f(x_0)+f(x_1)] = \frac{3}{2}.9 = 13.5$$

$h = \frac{3}{2}$

$n = 2$

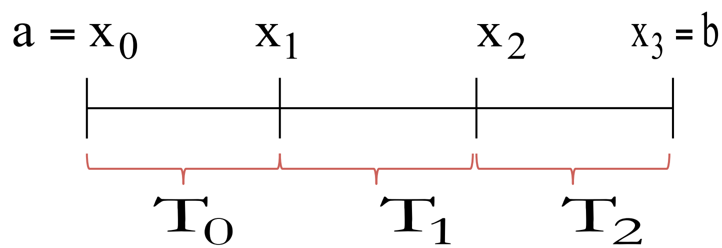


$$\tilde{I} = \frac{h}{2}[f(x_0)+2f(x_1)+f(x_2)] =$$

$$\frac{81}{8} = 10\ .1250$$

$h = 1$

$n = 3$

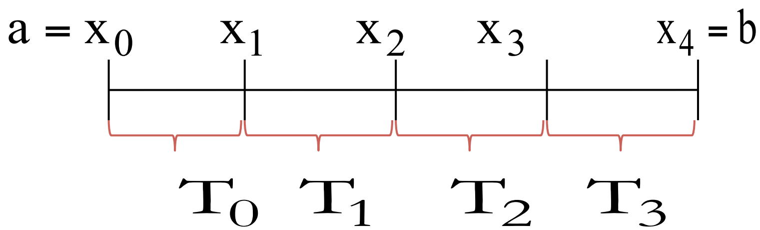


$$\tilde{I} = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)] =$$

$$\frac{19}{2} = 9.5$$

$h = \frac{3}{4}$

$n = 4$



$$\tilde{I} = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)] =$$

$$\frac{297}{32} = 9.28$$

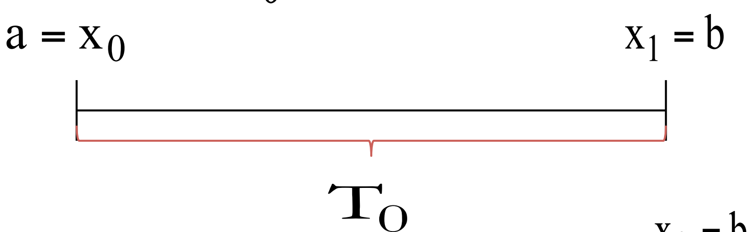
Esempio 4

TRAPEZI COMPOSTO

$$I = \int_0^1 \frac{1}{1+x}\,dx = 0.69314718$$

$h = 1$

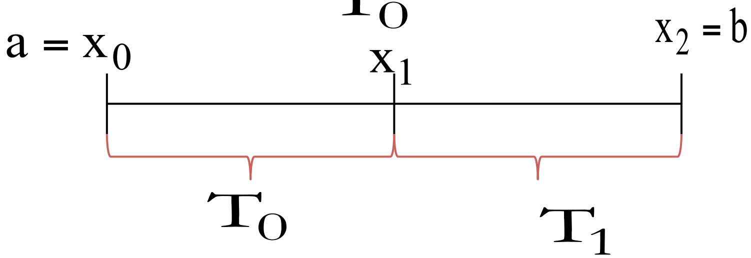
$n = 1$



$$\tilde{I} = \frac{h}{2}[f(x_0) + f(x_1)] = \frac{3}{4} = 0.75$$

$h = \frac{1}{2}$

$n = 2$

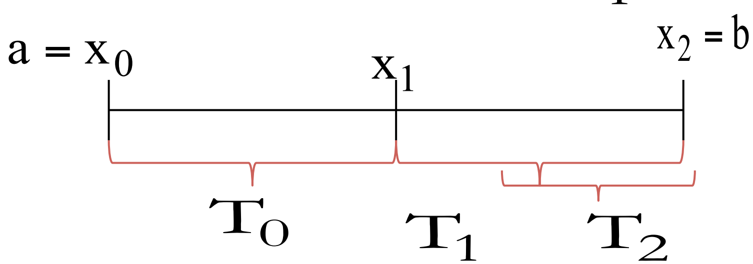


$$\tilde{I} = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)] =$$

$$\frac{17}{24} = 0.70833$$

$h = \frac{1}{3}$

$n = 3$

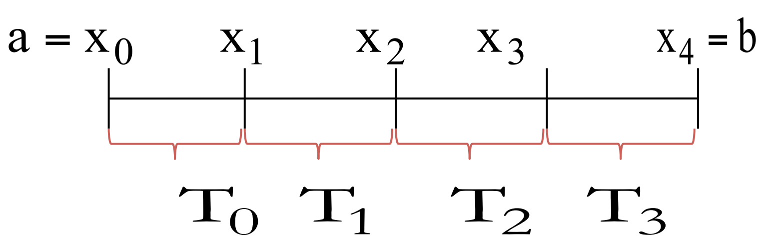


$$\tilde{I} = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)] =$$

$$\frac{21}{30} = 0.7$$

$h = \frac{1}{4}$

$n = 4$



$$\tilde{I} = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] =$$

$$\frac{1171}{1680} = 0.6970$$

$$I = \int_0^1 \frac{1}{1+x}\,dx = 0.69314718$$

	$\tilde{I}$	$\lvert \tilde{I} - I \rvert$
$\boxed{h = 1 \quad n = 1}$	0.75	0.0569
$\boxed{h = \frac{1}{2} \quad n = 2}$	0.70833	0.015
$\boxed{h = \frac{1}{3} \quad n = 3}$	0.70	0.069
$\boxed{h = \frac{1}{4} \quad n = 4}$	0.6970	0.0039

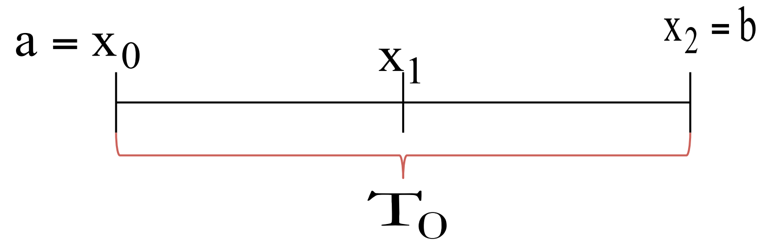
Metodo di Cavalieri-Simpson Composto

$\{x_0, x_1, \ldots, x_n\}$ **n pari** $T_k = [x_{2k}, x_{2k+1}, x_{2k+2}]$ $k = 0,1,\ldots,\frac{n}{2}-1$

Posto $m = \frac{n}{2}$

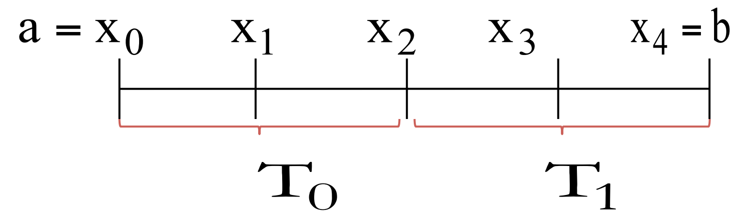
n=2
m=1

$k = 0$ $T_0 = [x_0, x_1, x_2]$



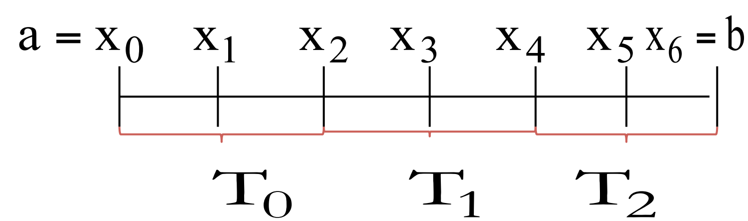
n=4
m=2

$k = 0,1$ $\begin{cases} T_0 = [x_0, x_1, x_2] \\ T_1 = [x_2, x_3, x_4] \end{cases}$



n=6
m=3

$k = 0,1,2$ $\begin{cases} T_0 = [x_0, x_1, x_2] \\ T_1 = [x_2, x_3, x_4] \\ T_2 = [x_4, x_5, x_6] \end{cases}$



Formula Generale

$$\tilde{I} = \frac{h}{3}\left[f(x_0) + 4\sum_{i=0}^{m-1} f(x_{2i+1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_n)\right]$$

Errore

$$R = \sum_{i=0}^{m-1} R_i \approx m\frac{-(b-a)5}{90n^5} f^{IV}(\xi) = -\frac{n}{2}\frac{(b-a)^5}{90n^5} f^{IV}(\xi) = \frac{-(b-a)^5}{180n^4} f^{IV}(\xi)$$

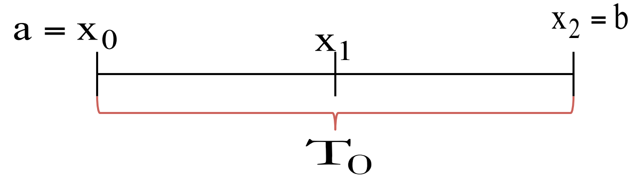
Esempio 5

Metodo di Cavalieri- Simpson Composto

$$I = \int_0^1 \frac{1}{1+x}\,dx = 0.69314718$$

h=1/2

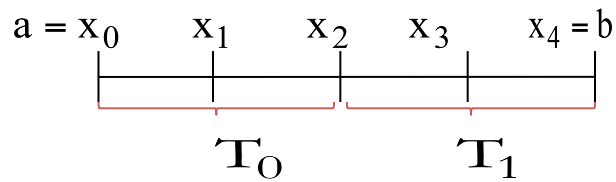
n=2
m=1



$$\tilde{I} = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x2)] = 0.6944$$

h=1/4

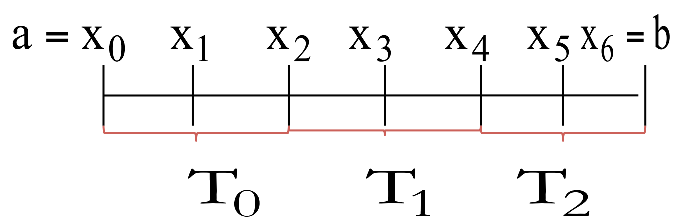
n=4
m=2



$$\tilde{I} = \frac{h}{3}[f(x_0) + 4f(x_1) + 4f(x_3) + 2f(x2) + f(x_4)] = 0.693253968$$

h=1/6

n=6
m=3



$$\tilde{I} = \frac{h}{3}[f(x_0) + 4f(x_1) + 4f(x_3) + 4f(x_5) + 2f(x2) + 2f(x_4) + f(x_6)] = 0.693169793$$

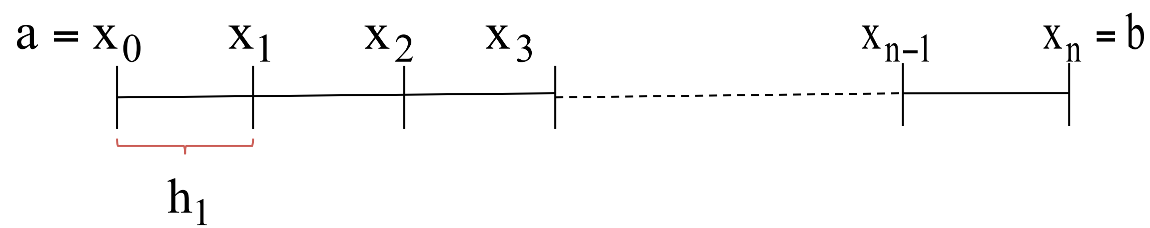
VALUTAZIONE INDIRETTA DEL RESTO

$\widetilde{I}_1$ **CALCOLATO CON FORMULA AVENTE GRADO DI PRECISIONE p**

$$h_1 = \frac{b-a}{n}$$

$$R_1 = f^{(p+1)}(\varepsilon_1)h_1^{p+2}\gamma$$

$$a < \varepsilon_1 < b$$

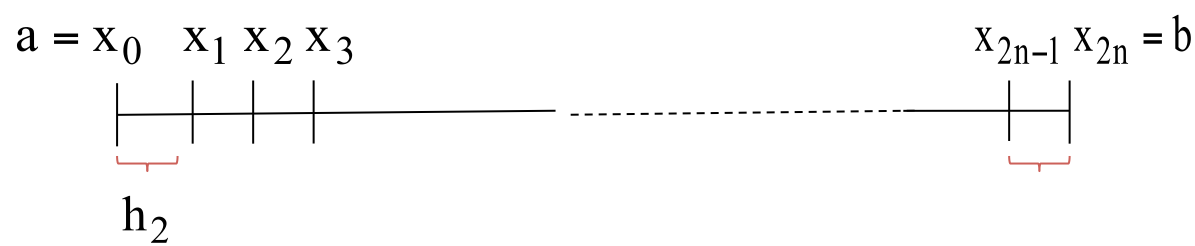


$\widetilde{I}_2$

$$h_2 = \frac{h_1}{2}$$

$$R_2 = f^{(p+1)}(\varepsilon_2)h_2^{p+2}2\gamma$$

$$a < \varepsilon_2 < b$$



$$I = \widetilde{I}_1 + R_1 \quad I = \widetilde{I}_2 + R_2 \qquad 0 = \widetilde{I}_2 + R_2 - \widetilde{I}_1 - R_1$$

$$\widetilde{I}_2 - \widetilde{I}_1 = R_1 - R_2 = f^{(p+1)}(\varepsilon_1)h_1^{p+2}\gamma - f^{(p+1)}(\varepsilon_2)h_2^{p+2}2\gamma =$$

$$f^{(p+1)}(\varepsilon_1)h_1^{p+2}\gamma - f^{(p+1)}(\varepsilon_2)\left(\frac{h_1}{2}\right)^{p+2}2\gamma \approx f^{(p+1)}(\varepsilon)h_1^{p+2}\gamma\frac{2}{2}\left[\frac{2^{p+1}-1}{2^{p+1}}\right] =$$

$$f^{(p+1)}(\varepsilon)h_1^{p+2}\gamma\frac{2}{2}\left[\frac{2^{p+1}-1}{2^{p+1}}\right] = f^{(p+1)}(\varepsilon)\left(\frac{h_1}{2}\right)^{p+2}2\gamma\left[2^{p+1}-1\right] = R_2\left[2^{p+1}-1\right]$$

$$\boxed{R_2 \approx \frac{\widetilde{I}_2 - \widetilde{I}_1}{2^{p+1}-1}}$$

ESTRAPOLAZIONE DI RICHARDSON

STIMA AUTOMATICA DEL RESTO $\quad |R_2| \approx \left| \dfrac{\tilde{I}_2 - \tilde{I}_1}{2^{p+1} - 1} \right| < \text{toll}$

Esempio 6

$$I = \int_0^1 e^{-x^2} dx = 0.7468241 \qquad R_2 \le 0.5 \cdot 10^{-3}$$

$a = x_0$ $\quad x_1 = b$ $\qquad h = b - a \qquad \tilde{I} = \frac{h}{2}[f(x_0) + f(x_1)]$

$a = x_0$ $\quad x_1$ $\quad x_2 = b$ $\qquad h = \frac{b-a}{2} \qquad \tilde{I} = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$

$a = x_0$ $\quad x_1$ $\quad x_2$ $\quad x_3$ $\quad x_4 = b$ $\qquad h = \frac{b-a}{4} \qquad \tilde{I} = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$

n	$\tilde{I}$
1	0.68393972
2	0.7313700
4	0.7429838
8	0.7458653
16	0.7465825

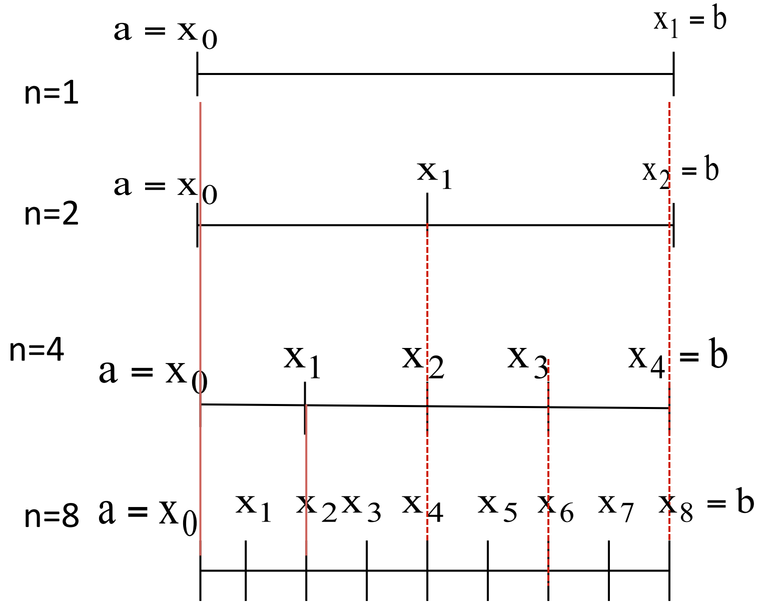
$R_2 \approx 0.01581009$

$R_2 \approx 0.00387127$

$R_2 \approx 0.0009615$

$R_2 \approx 0.00023907 \;\rightarrow\; R_2 \le 0.5 \cdot 10^{-3}$

METODO DI ROMBERG



$$h = b - a \qquad T_{11} = \tilde{I} = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$h = \frac{b-a}{2} \qquad T_{21} = \tilde{I} = \left(\frac{b-a}{2}\right)\left[\frac{f(a)+f(b)}{2} + f(a+h)\right]$$

$$h = \frac{b-a}{4} \qquad T_{31} = \tilde{I} = \left(\frac{b-a}{4}\right)\left[\frac{f(a)+f(b)}{2} + \sum_{r=1}^{3} f(a+rh)\right]$$

$$h = \frac{b-a}{8} \qquad T_{41} = \tilde{I} = \left(\frac{b-a}{8}\right)\left[\frac{f(a)+f(b)}{2} + \sum_{r=1}^{7} f(a+rh)\right]$$

$j = 1$	$j = 2$	$j = 3$	$j = 4$
T_{11}			
T_{21}	$T_{22} = T_{21} + R_2^{(22)}$		
T_{31}	$T_{32} = T_{31} + R_2^{(32)}$	$T_{33} = T_{32} + R_2^{(33)}$	
T_{41}	$T_{42} = T_{41} + R_2^{(42)}$	$T_{43} = T_{42} + R_2^{(43)}$	$T_{44} = T_{43} + R_2^{(44)}$
↓ **p=1**	↓ **p=3**	↓ **p=5**	↓ **p=7**

$$R_2^{(ij)} = \left(\frac{T_{i\,j\text{-}1} - T_{i\text{-}1\,j\text{-}1}}{2^{\underbrace{[2(j-1)-1]}_{p}+1} - 1} \right)$$

$n=2^s$ **Grado di precisione p=2j-1** **Ordine della matrice s+1**

$$R_2^{(ij)} = \left(\frac{T_{i\,j-1} - T_{i-1\,j-1}}{2^{\underbrace{[2(j-1)-1]}_{p}+1} - 1} \right)$$

Seconda colonna

$$T_{i2} = T_{i1} + R_2^{(i2)} = T_{i1} + \left(\frac{T_{i1} - T_{i-11}}{2^{p+1} - 1} \right)$$

p=2(j-1)-1=1

$$T_{i2} = T_{i1} + R_2^{(i2)} = \left(\frac{4T_{i1} - T_{i1} + T_{i1} - T_{i-11}}{4 - 1} \right)$$

$$\boxed{T_{i2} = \frac{4T_{i\,1} - T_{i-1\,1}}{4 - 1}}$$

Terza colonna

$$T_{i3} = T_{i2} + R_2^{(i3)} = T_{i2} + \left(\frac{T_{i2} - T_{i-12}}{2^{p+1} - 1} \right)$$

p=2(j-1)-1=3

$$T_{i3} = \frac{4^2 T_{i2} - T_{i2} + T_{i2} - T_{i-12}}{4^2 - 1}$$

$$\boxed{T_{i3} = \frac{4^2 T_{i\,2} - T_{i-1\,2}}{4^2 - 1}}$$

Formula Generale

$$\boxed{T_{ij} = \frac{4^{j-1} T_{i\,j-1} - T_{i-1\,j-1}}{4^{j-1} - 1} \qquad j = 2,\dots,s+1 \quad ; \quad i = j,\dots,s+1}$$

Esempio 7

$$I = \int_0^1 e^{-x^2} dx = 0.7468241$$

$$T_{11} = 0.68393972$$

$$T_{21} = 0.7313700 \quad T_{22} = 0.7471800$$

$$T_{31} = 0.7429832 \quad T_{32} = 0.7468550 \quad T_{33} = 0.7458333$$

$$T_{41} = 0.7458653 \quad T_{42} = 0.7468258 \quad T_{43} = 0.74682385 \quad T_{44} = 0.746823700$$

LINGUAGGIO DI PROGETTO

INPUT: a,b,m,f (N:B: $n^s = m = s+1$)

OUTPUT: T

1. h=b-a

2. c=[f(a)+f(b)] / 2

3. t_{11}=h c

4. som=0

5. n=2 % numero di intervalli

6. Ripetere per i=2,m

6. 1 h=h/2

6. 2 Ripetere per r=1,n/2

6. 2.1 som=som+f [a+(2r-1)h]

6. 3 t_{11}=h(c+som)

6. 4 n=2n

Prima colonna

7. r =1

8. Ripetere per j=2,m

8. 1 r=4r

8. 2 Ripetere per i=j, m

8. 2.1 $t_{i\,j} = (r\, t_{i\,j-1} - t_{i-1\;j-1})/(r-1)$