tp1.2

A continuing inquiry
into the Foundations
of the
Science of Physics

JRBreton

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by JRBreton

Published by:
The Foundation for Theoretical Physics
3 Apple Tree Lane
Walpole, MA 02081−2301

Web address: FoundationForTheoreticalPhysics.org
email address: theoretical.physics.books@gmail.com

Copies of this book and other offerings of the Foundation may be obtained preferably online from tp.vendevor.com, the Foundation's website and also from amazon.com and similar sites.

ISBN, print ed. **978−0−9844299−7−4**

First printing 2019

Printed in the United States of America

Library of Congress Control Number **2016933089**
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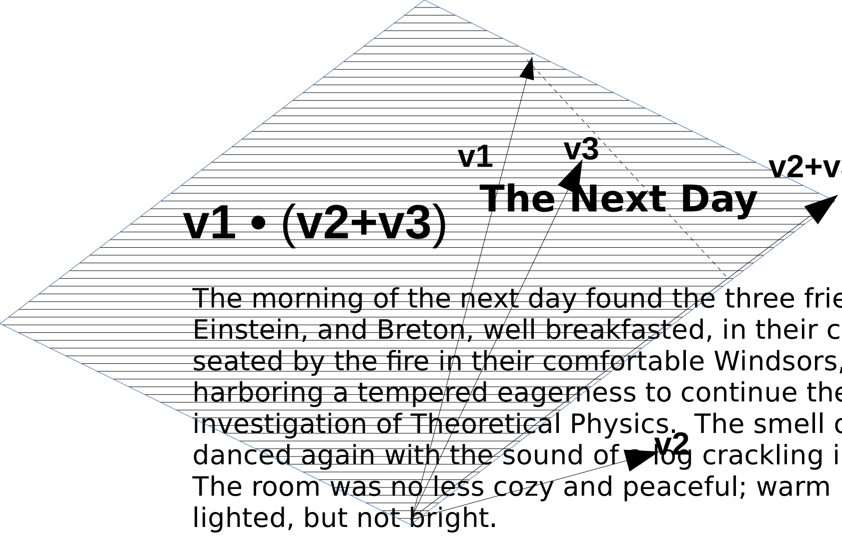

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The Next Day

The morning of the next day found the three friends, Newton, Einstein, and Breton, well breakfasted, in their clubhouse, seated by the fire in their comfortable Windsors, each harboring a tempered eagerness to continue their investigation of Theoretical Physics. The smell of Autumn danced again with the sound of a log crackling in the fireplace. The room was no less cozy and peaceful; warm but not hot; lighted, but not bright.

Newton, who loved to summarize and recapitulate as well as construct tables, began the conversation with an attempt to summarize the conclusions of the previous day. “We discussed so many items yesterday, let us start by summarizing yesterday's conclusions. We came to see how a science like Physics differs from a technology like Surveying.

Breton, interrupting: “Technology is useful, and when no longer useful, discarded. It willingly sacrifices truth for utility. Science is not necessarily useful but will not compromise truth and so is permanent.

Einstein, adding in his friendliest tone: “Technology relies on measurement, but science does not.

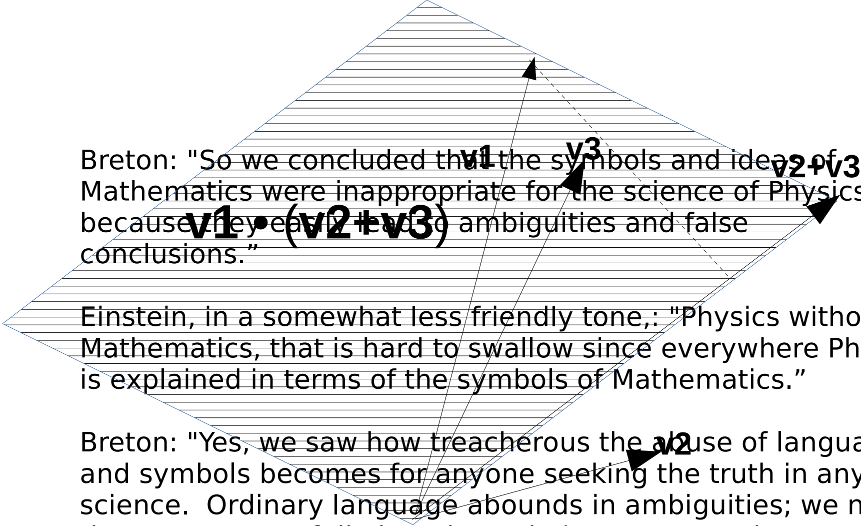
Newton: ”And many more differences which you will remember from my tables of yesterday, but sciences differ from each other too. The science of Physics differs from the science of Mathematics.

Einstein: "They are both true and permanent, but differ in what they are true to.

Breton: "Mathematics is true to its axioms while Physics must be true to some aspect of reality.

Newton: “We accepted, after some debate the following definition of Physics:

> **Physics** is the study of reality
> observable as extended, moving, or forcing.



Breton: "So we concluded that the symbols and ideas of Mathematics were inappropriate for the science of Physics because they easily lead to ambiguities and false conclusions."

Einstein, in a somewhat less friendly tone,: "Physics without Mathematics, that is hard to swallow since everywhere Physics is explained in terms of the symbols of Mathematics."

Breton: "Yes, we saw how treacherous the abuse of language and symbols becomes for anyone seeking the truth in any science. Ordinary language abounds in ambiguities; we noted the emergence of dictionaries to help us communicate.
Since truth in general and scientific truth in particular cannot tolerate contradictions, sciences are forced to construct special dictionaries to avoid ambiguities in their disciplines. Mathematics has its dictionary; Physics should also have its own, distinct from the one for Mathematics. We call the dictionary for the science of Physics by the name Theoretical Physics. It, not Mathematics, is the proper language for Physics."

Einstein: "But for all that, the language of Mathematics does seem appropriate for the study of Physics."

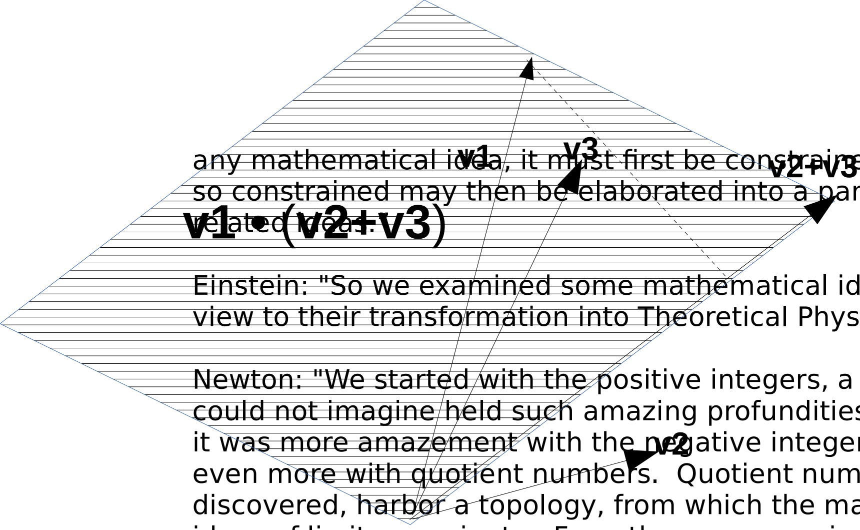
Breton: "Mathematics has great appeal because of its simple, logical structure, a quality which should also characterize an appropriate language for Physics. Nevertheless the ideas of Mathematics are not the ideas of Theoretical Physics, even were they to use the same symbols. In Mathematics, a symbol would refer to the mathematical dictionary, while the same symbol in Theoretical Physics would refer to a different dictionary. A grave confusion results from using the wrong dictionary."

Newton: "We illustrated all this. For instance, the proposition

$$2+2=4$$

is true enough as a mathematical statement, but ambiguous or even false as a physical statement."

Breton: "So we embarked on a great adventure to discover how mathematical ideas and propositions can be transformed into ideas and propositions suitable for Theoretical Physics. The outlines of the adventure are clear enough: to transform



any mathematical idea, it must first be constrained, and when so constrained may then be elaborated into a panoply of related ideas.

Einstein: "So we examined some mathematical ideas with a view to their transformation into Theoretical Physics.”

Newton: "We started with the positive integers, a subject I could not imagine held such amazing profundities. From there it was more amazement with the negative integers and then even more with quotient numbers. Quotient numbers, we discovered, harbor a topology, from which the mathematical ideas of limits germinate. From there we examined the amazing world of functions, and ideas of continuity, derivatives and integrals.”

Breton: "You've omitted so much.”

Newton: "True enough: topics like look‒alike functions, and step functions, and many others besides. When I reflect on our conversation yesterday I stand amazed at the amazing topics we wrestled with. A brief summary just omits too much. Yesterday's conversation should be made into a book!”

Breton: "What would be its title?”

Newton: "Why don't *you* propose a title?”

Breton: "Let's title it 'tp1.1'. The title would stand for theoretical physics 1.1. The 1.1 would indicate more to come.”

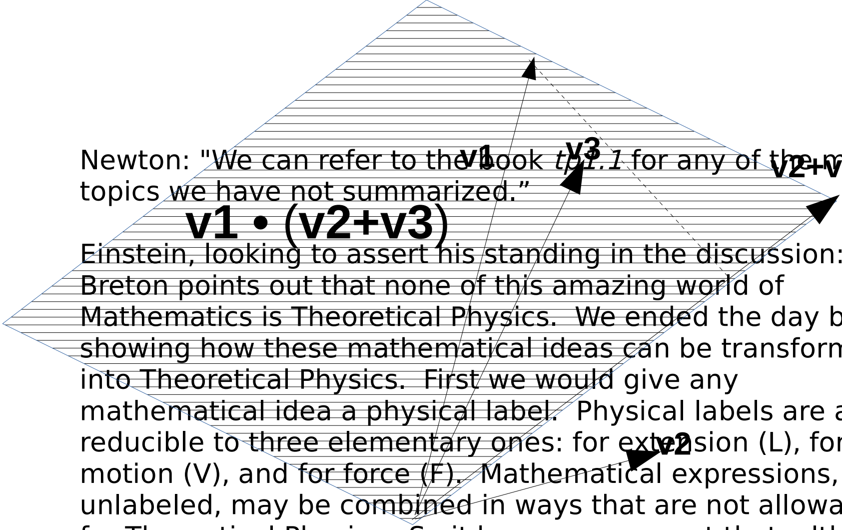
Einstein: "Who, except us, would know what tp1.1 means?”

Breton: "We could give it a subtitle like 'an inquiry into the foundations of the science of Physics.'”

Einstein: "Better, but still obscure.”

Breton: "Would *you* like to try your hand at a title?”

Einstein, after a pause: "No. The book might appeal to adventurous and inquiring minds and surely discourage shallow, superficial surfing. Let the title stand.”



Newton: "We can refer to the book *tpi.1* for any of the many topics we have not summarized."

Einstein, looking to assert his standing in the discussion: "But Breton points out that none of this amazing world of Mathematics is Theoretical Physics. We ended the day by showing how these mathematical ideas can be transformed into Theoretical Physics. First we would give any mathematical idea a physical label. Physical labels are all reducible to three elementary ones: for extension (L), for motion (V), and for force (F). Mathematical expressions, being unlabeled, may be combined in ways that are not allowable for Theoretical Physics. So it became apparent that, although an identical symbol might be used, a number is mathematics is different idea from a number in Theoretical Physics."

Newton, continuing: "Expressions in Theoretical Physics must follow the Rules for Labels. The Rules show how new ideas for Theoretical Physics can be created from the elementary ones."

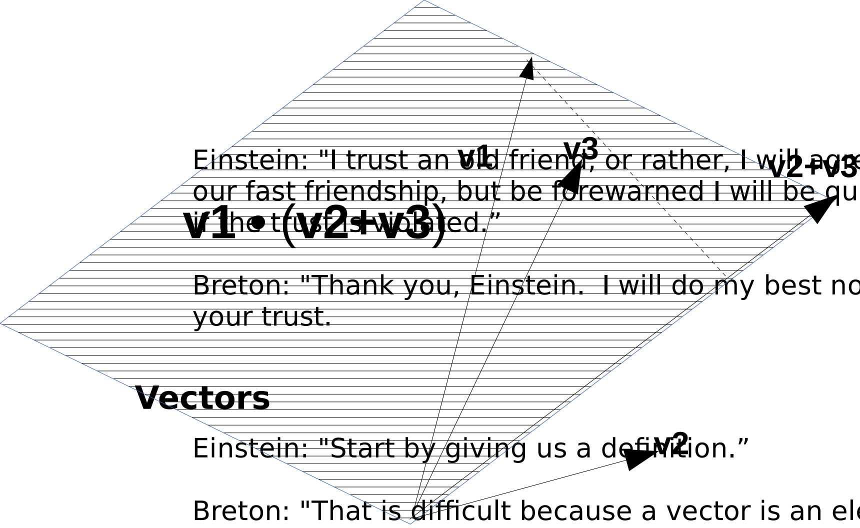
Einstein, still looking to lead: "In addition to labeling, we saw how the ideas of Theoretical Physics must be referenced either materially or spatially."

Breton: "And how the word 'set' can said of material things as well as mathematical ideas which led to the idea of a particle, the properties of material things, and the constraints of resolution."

Einstein: "We would do well to deepen our conversation about these topics since they promise application to the science of Physics."

Newton: "But Breton suggested that today we look into the subject of location."

Breton, looking to smooth the rising contention between his friends: "Thank you Newton. We observe physical objects located here and there. Yet very little of what we discussed yesterday faced this aspect of physical reality. Mathematics provides an interesting structure called *vectors* which may be suitable for transformation into Theoretical Physics. I suspect we will deepen our knowledge of yesterday's topics by seeing them in this new perspective. Will you trust me, Einstein?"



Einstein: "I trust an old friend, or rather, I will agree based on our fast friendship, but be forewarned I will be quick to object if the trust is violated.”

Breton: "Thank you, Einstein. I will do my best not to violate your trust.

Vectors

Einstein: "Start by giving us a definition.”

Breton: "That is difficult because a vector is an elemental idea. There aren't many antecedents upon which I can build a definition. For instance, if I defined a vector to be an element in a vector space you would say immediately that that defines nothing.”

Einstein, with not a little sarcasm: "Without a definition we don't know what we are talking about.”

Breton: "Agreed. What elemental experience can I refer to for a start?”

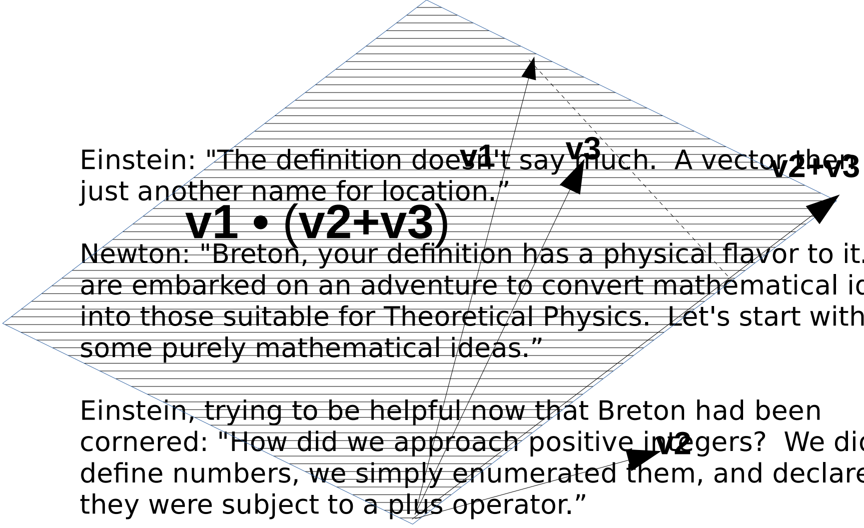
Newton: "you noted yesterday that of the many topics we discussed, nothing touched location. Yet Physics deals with extension, motion and force, all of which imply a location at which an object can be observed as extended, moving, or forcing. So let me suggest we take location as a given upon which to build a definition.”

Breton: "Let's try this definition.

Definition (vector)
Given
the location of an object
then

a **vector** is an idea which specifies its location.

end of definition



Einstein: "The definition doesn't say much. A vector then is just another name for location."

Newton: "Breton, your definition has a physical flavor to it. We are embarked on an adventure to convert mathematical ideas into those suitable for Theoretical Physics. Let's start with some purely mathematical ideas."

Einstein, trying to be helpful now that Breton had been cornered: "How did we approach positive integers? We did not define numbers, we simply enumerated them, and declared they were subject to a plus operator."

Newton: "Or alternatively we declared the positive integers to be the result of an indefinite application of the plus operator on a number called one."

Breton: "So, we should be looking for axioms, rather than definitions?"

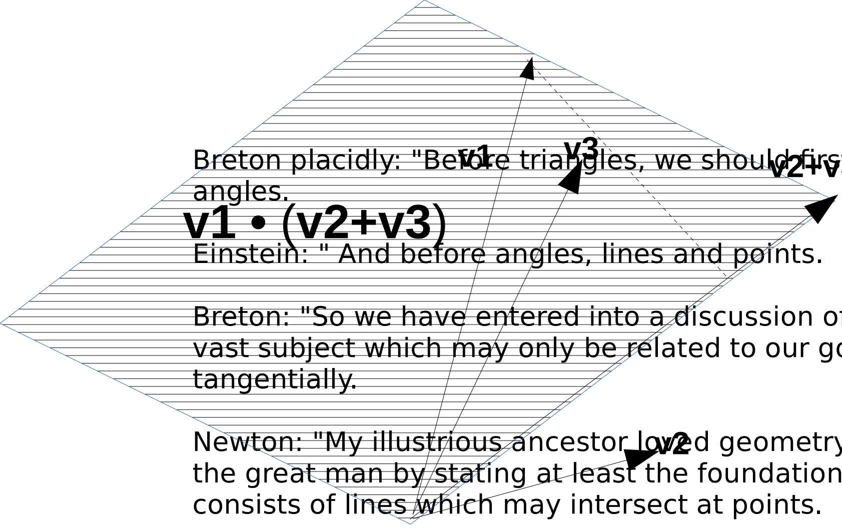
Einstein: "What's the difference?

Breton: "Axioms are fundamental statements upon which a logical structure can be erected. Like rules for a game, they need to be simply accepted. If the axioms are changed a different structure will emerge. Think of Euclid's axioms for geometry. They form a basis for a plane geometry. Change the axioms, a new geometry will appear.
   Definitions are built upon the axioms. They use the accepted axioms including their terms as a root vocabulary."

Newton: "How does this fit in with location?"

Breton: "Location is an attribute of an object. If the object is a material one, location is a physical attribute, not an idea at all. A vector is an *idea* which hopefully can be transformed to describe a location. To provide all possible descriptions for locations we create a set ideas of all possible lengths and all possible directions."

Einstein, looking to derail a coming argument: "But what if the object is a mathematical idea like a triangle?"



Breton placidly: "Before triangles, we should first discuss angles.

Einstein: " And before angles, lines and points.

Breton: "So we have entered into a discussion of geometry, a vast subject which may only be related to our goal tangentially.

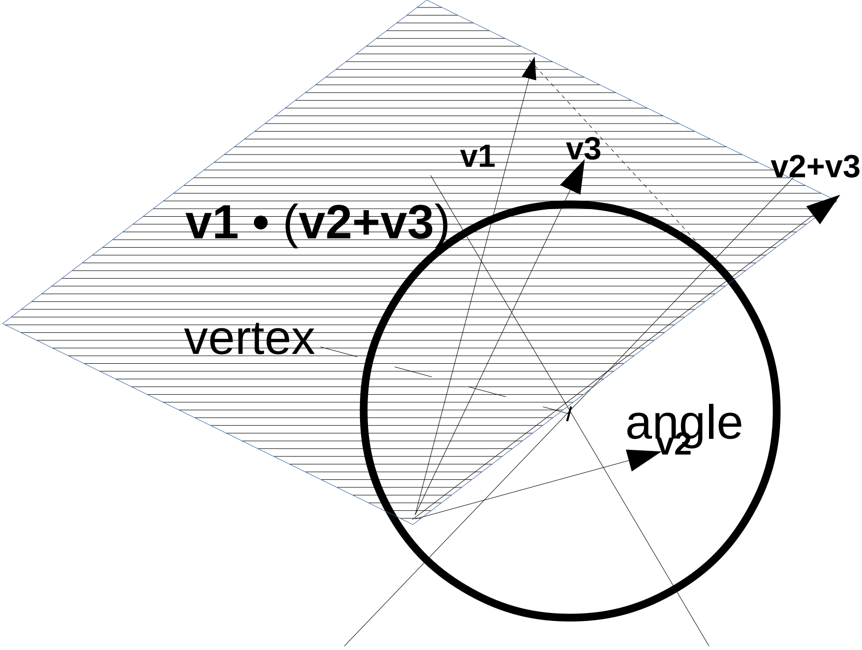
Newton: "My illustrious ancestor loved geometry. Let us honor the great man by stating at least the foundations. Geometry consists of lines which may intersect at points.

Breton, trying to angle a return towards the main goal. “The point at which two lines intersect may be called a **vertex**. At the vertex four **angles** are formed between the lines.
“There are many kinds of angles; it will be worth our while to define them and then consider how they apply to triangles.

Einstein, happily interrupting,: “And how do you measure angles?”

Breton: "You bring up another good point. Angles, indeed, can be measured because they have parts. As a mathematical idea, an angle is complex. We started with two lines which intersect. The intersection, called the **vertex**, can form the center of a circle. Further, we can truncate one of the lines finitely at the vertex and let it be the radius of a circle. An **angle** is this complex of lines, vertex and circle. To measure the angle, note that the two lines, intersecting the circle, define an arc of the circle. The ratio of the length of the arc compared to the circumference of the circle is used to measure the angle.”

With that Breton quickly sketched this illustration.

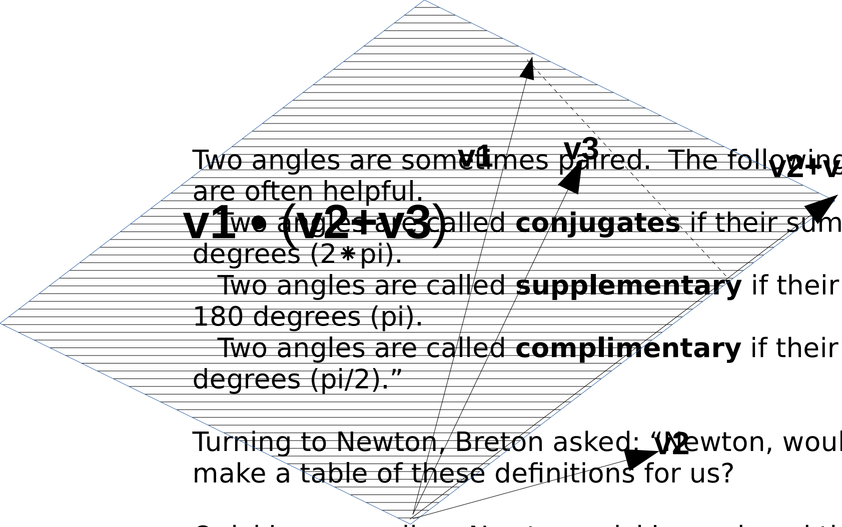


Newton: "So different measures result from the measurement of the circumference.

Breton: "Exactly. Two of the most common are called **measurement in degrees** and **measurement in radians**. For measurement in degrees the circumference is divided into 360 equal parts. The arc of the angle will then be measured as so many of these degrees. For measurement in radians the circumference is divided into the number of radii which will fit into it. That number is 2*pi. The arc of the angle will then be measured as so many of these radii, called **radians**.

Einstein: "So the actual measurement is accomplished in terms of arbitrary units.

Breton: "Not arbitrary. The measurement assumes a reference which must be stated but often merely assumed when the unit is declared.
If all this is clear, let us return to the definitions of different angles.
...An **acute** angle is an angle less than 90 degrees (pi/2);
...a **right** angle is an angle equal to 90 degrees (pi/2);
...an **obtuse** angle is an angle greater than 90 degrees (pi/2), but less than 180 degrees (pi);
...a **reflex** angle is an angle greater than 180 degrees (pi).



Two angles are sometimes paired. The following definitions are often helpful.

Two angles are called **conjugates** if their sum equals 360 degrees (2*pi).

Two angles are called **supplementary** if their sum equals 180 degrees (pi).

Two angles are called **complimentary** if their sum equals 90 degrees (pi/2)."

Turning to Newton, Breton asked: "Newton, would you please make a table of these definitions for us?

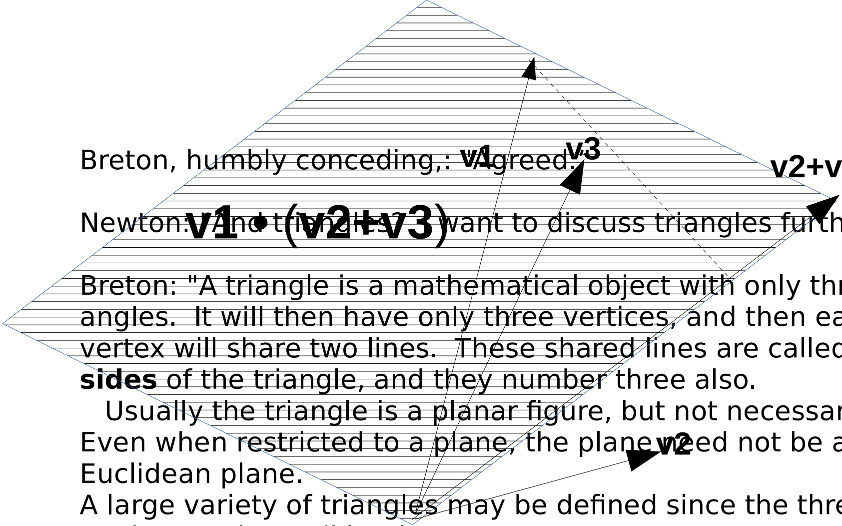
Quickly responding, Newton quickly produced the following table.

Angles	
Type	**Definition**
acute	Less than pi/2
right	Equal to pi/2
obtuse	Greater than pi/2
reflex	Greater than pi
Conjugate	Sum equals 2*pi
Supplementary	Sum equals pi
Complimentary	Sum equals pi/2

Einstein: "Then a **triangle** is a mathematical structure of lines forming three angles.

Breton, returning to the main track gleefully: "Then the *location* of one vertex can be referred to a second vertex. In a similar way the location of a physical object must be referred to some observer. You bring up some good points Einstein.
In surveying, mathematical triangles play no unimportant role. Triangles are not numbers. Would it be worthwhile to begin our study of location and vectors with triangles?

Einstein, needling Breton: "Since we are searching for foundations, angles would be a better choice. Don't you agree angles are more basic than triangles?



Breton, humbly conceding,: "Agreed."

Newton: "And triangles? I want to discuss triangles further.”

Breton: "A triangle is a mathematical object with only three angles. It will then have only three vertices, and then each vertex will share two lines. These shared lines are called **sides** of the triangle, and they number three also.

Usually the triangle is a planar figure, but not necessarily so. Even when restricted to a plane, the plane need not be a Euclidean plane.

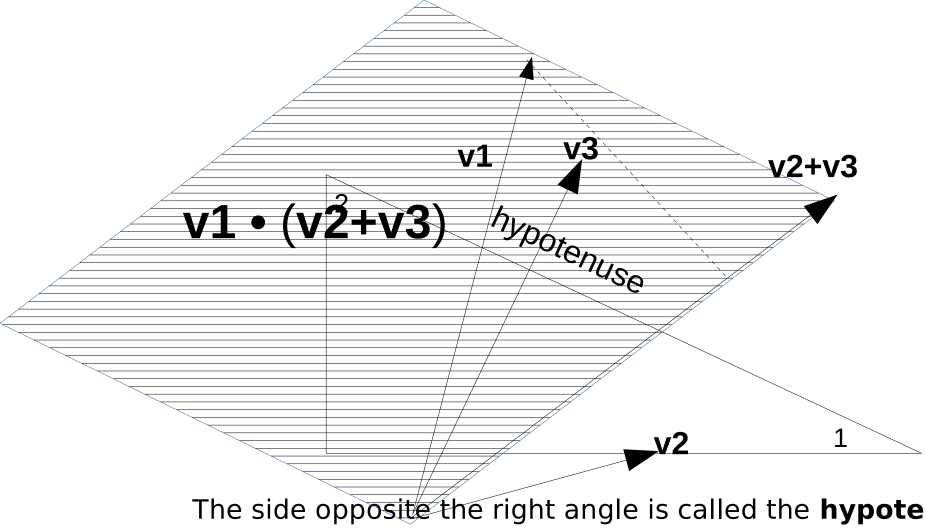
A large variety of triangles may be defined since the three angles need not all be the same.

...An **oblique** triangle is one all of whose angles are acute;
...a **right** triangle is one which has one right angle.
...an **equilateral** triangle is one all of whose angles are equal;
...an **isosceles** triangle is one with two equal angles;
...a **scalene** triangle is one none of whose angles are equal.

Newton, anticipating a request quickly produced the following table without being asked.

Triangles	
Type	**Definition**
oblique	All angles acute
right	One right angle
Equilateral	All angles equal
isosceles	Two angles equal
scalene	No angles equal

Breton: "A couple of special definitions associated with right triangles should be noted. First let me sketch a right triangle.



The side opposite the right angle is called the **hypotenuse**. The other two sides are orthogonal to each other. The following two definitions should be remembered.

sin(angle) ≡ length of the side opposite the angle
/length of the hypotenuse

cos(angle) ≡ length of the side nearest the angle
/length of the hypotenuse

As you can see from the sketch

sin(angle1) = cos(angle2)

and

sin(angle2) = cos(angle1)

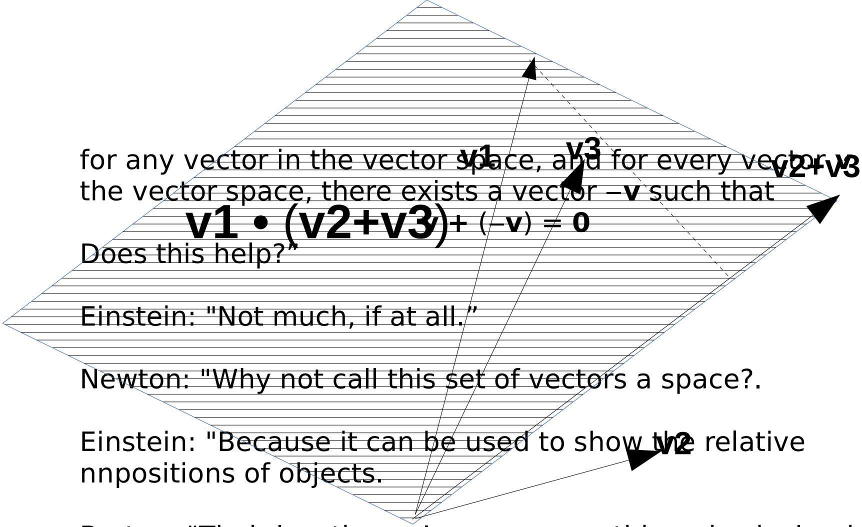
Einstein, taking charge of the discussion again: "Just give us a definition of a vector.

Breton: "Rather let me give you the *axioms* of a mathematical set of vectors which we may symbolized as **V**. The space is populated by elements called vectors symbolized by **v**. Our space of vectors presupposes the set of quotient numbers Q with its algebra and topology. The axioms also presuppose two operations, vector addition (symbolized by **+**) and multiplication by quotient numbers called **scalar** multiplication (symbolized by ∗) which adhere to the following axioms:

v1 + (v2 + v3) = (**v1 + v2**) **+ v3**:
v1 + v2 = **v2 + v1**
q1∗(q2∗**v1**) = (q1∗q2)∗**v1**
q1∗(**v1 + v2**) = q1∗**v1 +** q1∗**v2**
(q1+q2)∗**v1** = q1∗**v1 +** q2∗**v1**
1∗**v** = **v**, for any vector in the vector space.

Also, there exists a zero vector in the vector space symbolized by **0** such that

v+ 0 = **v**



for any vector in the vector space, and for every vector **v** in the vector space, there exists a vector −**v** such that

v + (−**v**) = **0**

Does this help?"

Einstein: "Not much, if at all."

Newton: "Why not call this set of vectors a space?.

Einstein: "Because it can be used to show the relative nnpositions of objects.

Breton: "Their locations. Is space something physical or is it a mathematical idea?

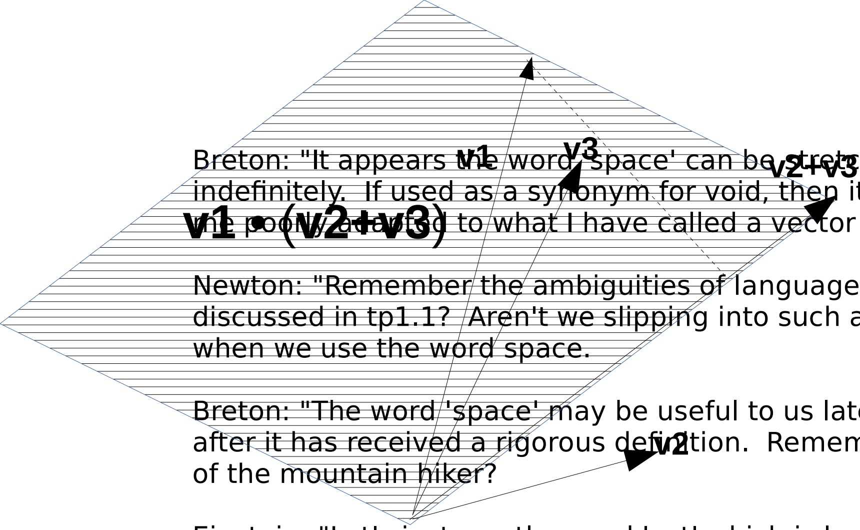
Einstein: "Physical!

Newton: "Mathematical!

Breton: "Your answers show that this question should be addressed. It seems to me that some uses of the word 'space' are physical or quasi−physical, and others mathematical. Let me list some current uses of the word:

outer space, as the universe beyond earth's atmosphere
as a gap between written characters. ASCII code 32.
personal space in human relationships.
a square in a board game.
a business term to describe a competitive environment
a solution space, candidates for solutions of equations
mental space in cognitive science
a vacuum
some buildings
address space in computers
cyberspace
white−space as allocated but locally unused radio frequencies

Newton: "Enough. The word 'space' can be used in science and engineering, but also in fiction, music, art, law and many, many other contexts.



Breton: "It appears the word 'space' can be stretched indefinitely. If used as a synonym for void, then it seems to me poorly adapted to what I have called a vector set.

Newton: "Remember the ambiguities of language we discussed in tp1.1? Aren't we slipping into such a morass when we use the word space.

Breton: "The word 'space' may be useful to us later, but only after it has received a rigorous definition. Remember the story of the mountain hiker?

Einstein: "Let's just use the word 'set' which is less ambiguous and can refer to mathematical objects unequivocally.

Breton: "Fine. So we can look to examine vector sets. But let us reflect on the intellectual path we have covered. First we thought to define the set V

$$\mathbf{V} \equiv \{\mathbf{v}\}$$

as a set of objects. Einstein rightly remarked this definition said little.
Next, we specified further

$$\mathbf{V} \equiv \{\{\mathbf{v}\},Q,\mathbf{+},\ast\}$$

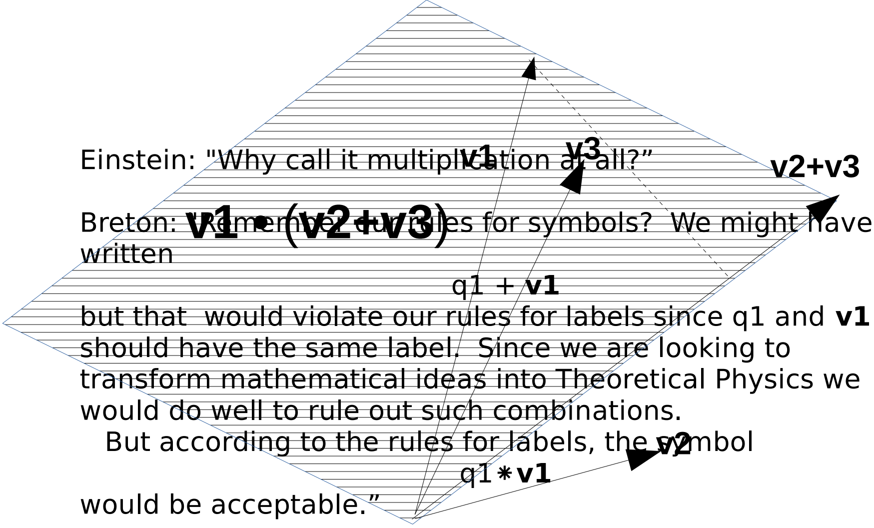
where Q is our algebra of quotient numbers, **+** a new kind of addition, and ⁕ a new kind of multiplication. While we have defined Q previously in tp1.1, the two operators remain unspecified.

Newton, reflectively: "Look! The axioms for the set of vectors have extended Q. If we take the vectors to be the partitions of the quotient numbers, and plus and multiply as defined for Q, then Q is itself a vector space."

Breton: "We are well started then. The set of vectors will be a set rooted in Q, associated with it by scalar multiplication, but possibly developed far beyond Q. So if **v** is a vector and q1 is a quotient number, then q1⁕**v** is also a vector.

Einstein, objecting: "Hold it there. This is a strange multiplication different from any of the others we seen."

Breton: "True enough. Very little gets by you, Einstein. This is yet another kind of multiplication, a multiplication of a different color."



Einstein: "Why call it multiplication at all?”

Breton: "Remember our rules for symbols? We might have written

q1 + **v1**

but that would violate our rules for labels since q1 and **v1** should have the same label. Since we are looking to transform mathematical ideas into Theoretical Physics we would do well to rule out such combinations.

But according to the rules for labels, the symbol

q1∗**v1**

would be acceptable.”

Einstein: "But why call it multiplication at all?"

Breton: "Mathematicians call it **scalar** multiplication, a special kind declared for vector sets. The word scalar is used because operations like

q1∗q2∗**v1**

are allowed. This kind of operation 'scales' the vector by making it longer of shorter.”

Einstein: "Now you're mixing the multiplications. The first applies only to Q; the second only to the set of vectors. They must be different operators.”

Breton: "Right again. They *are different* operators and must be seen as such. For his reason I have emboldened the second symbol for multiplication. I ask you to tolerate this admitted confusion for simplicity's sake. Scalar multiplication is stipulated by the axioms of a vector space.”

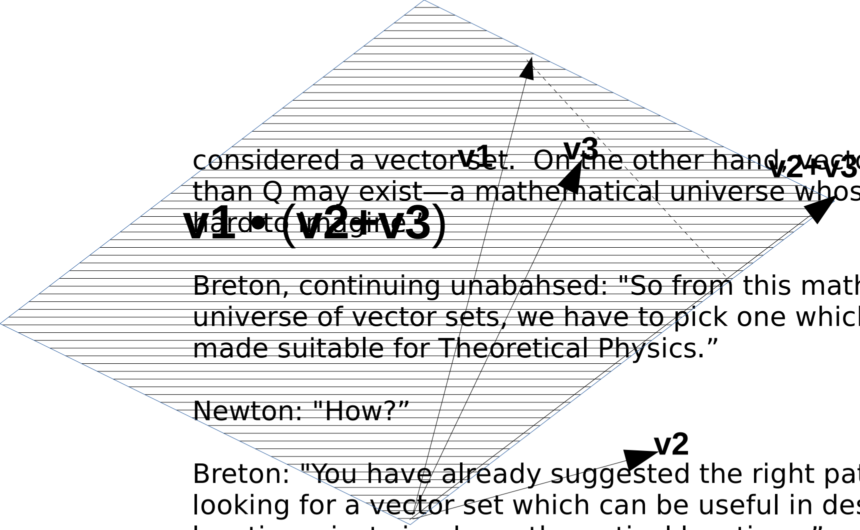
Einstein, thinking to corner Breton again: "This could get confusing.”

Breton, naively: "It gets worse. Mathematicians also endow the vector set with some algebraic features. So the set of vectors has its own plus operator such that if **v1** and **v2** are vectors then

v1 + v2

is also a vector.

Einstein, now looking to scope out the problem: "We know about Q, and as Newton has observed, by itself it can be



considered a vector set. On the other hand, vector sets other than Q may exist—a mathematical universe whose extent is hard to imagine.

Breton, continuing unabahsed: "So from this mathematical universe of vector sets, we have to pick one which can be made suitable for Theoretical Physics."

Newton: "How?"

Breton: "You have already suggested the right path. We are looking for a vector set which can be useful in describing locations, just simple mathematical locations."

Einstein, scornfully: "What is a simple, mathematical location?"

Newton: "That's easy enough. In conformity to its axioms, the vector set has a zero vector. A simple mathematical location is the vector related to the zero vector. For instance, my illustrious ancestor, could describe the position of a planet with relation to the sun as a vector relative to the zero vector taken as the sun."

Breton, noting a flashing from Einstein's eyes, pressed on quickly to avoid a confrontation: "So what is needed is a vector which incorporates both a distance and a direction. The distance can be taken as a scalar from Q, the quotient numbers, but what about direction?"

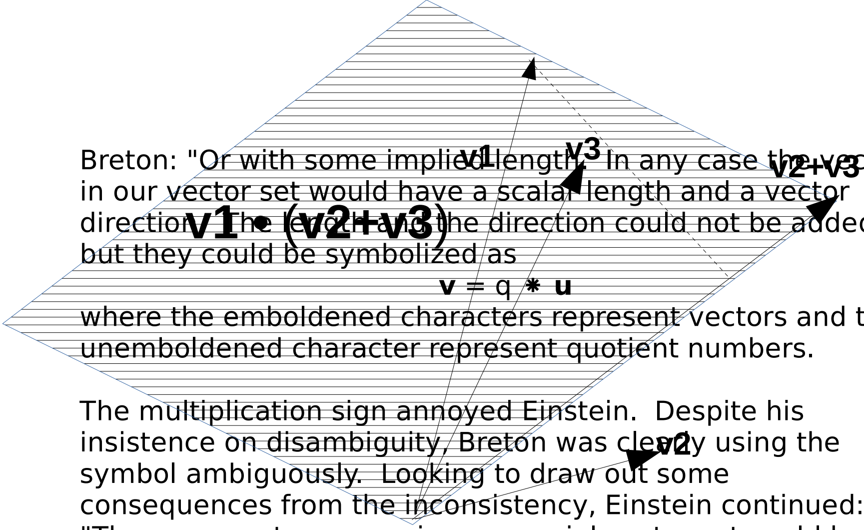
Einstein, stubbornly: "I don't like the word 'distance'. It connotes an idea which relates more to physical reality. Let's use a more mathematical word like 'length'."

Breton, in a conciliatory mood: "Fine. So let every vector in our vector set be composed of a length and a direction."

Newton, helpfully: "Direction can be thought of as points on a sphere centered on the zero vector."

Breton, probing: "So can directions be taken from Q?"

Newton, with a touch of irritation: "No, directions seem to be some kind of vector themselves without length."



Breton: "Or with some implied length. In any case the vectors in our vector set would have a scalar length and a vector direction. The length and the direction could not be added, but they could be symbolized as

$$\mathbf{v} = q * \mathbf{u}$$

where the emboldened characters represent vectors and the unemboldened character represent quotient numbers.

The multiplication sign annoyed Einstein. Despite his insistence on disambiguity, Breton was clearly using the symbol ambiguously. Looking to draw out some consequences from the inconsistency, Einstein continued: "Then any vector, say **v,** in our special vector set could be written as

$$\mathbf{v} = q(\mathbf{v}) * \mathbf{u}(\mathbf{v})$$

Newton, seizing the argument with a certain bustle: "If

$$\mathbf{v} = \mathbf{u}(\mathbf{v})$$

then q(**v**) must equal one.

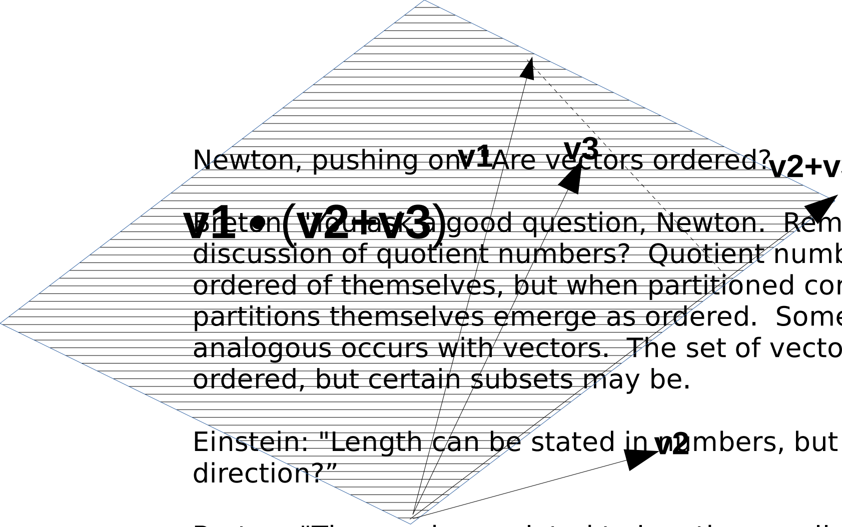
Breton, happily concluding: "So direction is a vector whose length is one. We might call such vectors **unit vectors**.”

Einstein, somewhat miffed because the argument had not gone as he expected: "I suspect you anticipated all this by symbolizing '**u**' for directions since they are unit vectors.”

Newton, ignoring Einstein's ignoble suggestion, wrapped up the conclusion: "Then directions are simply unit vectors, one for each point on a unit sphere centered on the zero vector.”

Einstein: "Length and direction are measurable. Have we fallen from science into technology? Remember we agreed that technology relies on measurement, a trait that separates technology from science.”

Breton, patiently: "Measurements result in numbers; vectors are not numbers. Recall our previous discussion. Theoretical Physics deals with objects that are measurable, because extended. Relationships between its ideas can be explained without actually measuring anything, just as relationships between mathematical ideas can be explained without measurements.



Newton, pushing on: "Are vectors ordered?

Breton: "You ask a good question, Newton. Remember our discussion of quotient numbers? Quotient number are not ordered of themselves, but when partitioned correctly, the partitions themselves emerge as ordered. Something analogous occurs with vectors. The set of vectors is not ordered, but certain subsets may be.

Einstein: "Length can be stated in numbers, but what about direction?"

Breton: "The numbers related to length are called the **underlying field** of the vector and for Theoretical Physics the set of partitions of quotient numbers, called Q, is sufficient. Many mathematicians, however, prefer to use the real numbers, R, as the underlying field."

Einstein: "I prefer to say 'completed numbers', rather than real numbers."

Breton: "As you will. I mean to emphasize that the underlying field comes with its algebra and topology."

Einstein: "How about direction?

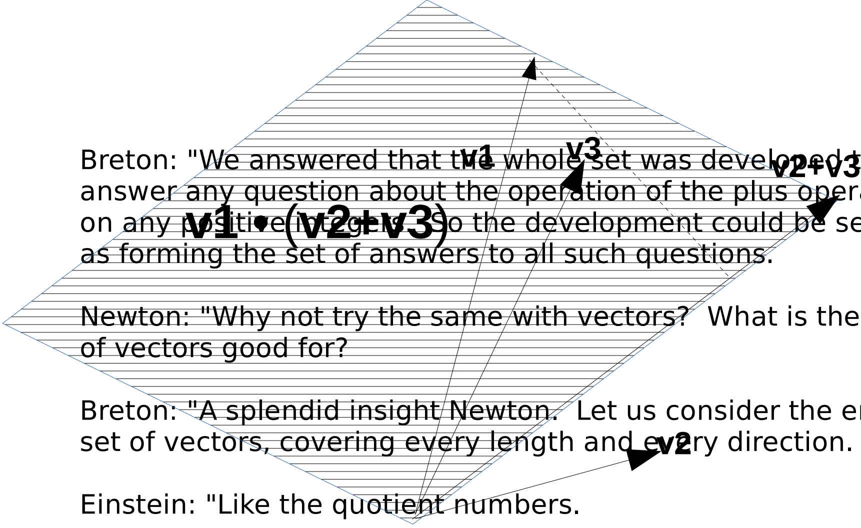
Breton: "Imagine you are standing in the center of a sphere. Any point on the sphere from your perspective would be a direction."

Newton: "So these imaginings give us some idea of a vector, but one I find little helpful. Let me suggest another approach. Remember how the positive integers were developed. Yesterday, Breton asked me for not one integer, but the whole set of them."

Einstein: "Then I proposed how the whole set could be generated by an algorithm."

Breton: "And one of you, I don't remember which, questioned why we should develop the whole set at all.

Einstein: "I did."



Breton: "We answered that the whole set was developed to answer any question about the operation of the plus operator on any positive integers. So the development could be seen as forming the set of answers to all such questions.

Newton: "Why not try the same with vectors? What is the set of vectors good for?

Breton: "A splendid insight Newton. Let us consider the entire set of vectors, covering every length and every direction.

Einstein: "Like the quotient numbers.

Breton: "Something like, but not the same. The directions of vectors refer to a sphere, while the directions of quotient numbers refer to a circle.

Einstein: "So the whole set of vectors provide answers about locations.”

Newton: "Vectors are growing a little more useful, but not much.

Einstein: "You represent direction as a vector. Isn't direction an angle?

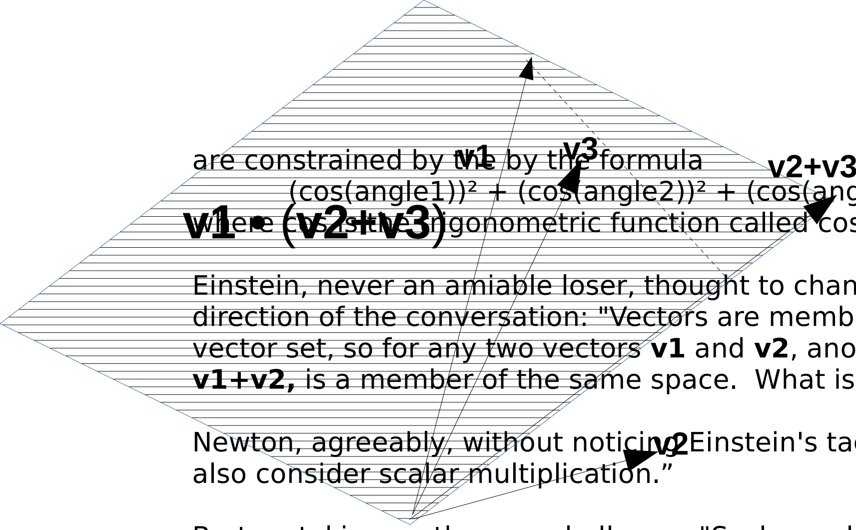
Breton: "You ask an interesting question. Which is more fundamental: angle or direction?

Einstein: "Angle because directions are stated in terms of angles.

Breton: "But angles need a reference. Isn't an angle measured between two directions? So it seems to me that direction is the more fundamental concept. I can point to something to show its direction without any reference to an angle.

Einstein, doggedly: "In any case angle and direction are closely related ideas.

Breton: "Different, but related. Moreover, once a system of axes is accepted, any point on the sphere can be located by three angles. If the three angles are called angle1, angle2, and angle3, then the three angles defining a given direction



are constrained by the by the formula

$$(\cos(\text{angle1}))^2 + (\cos(\text{angle2}))^2 + (\cos(\text{angle3}))^2 = 1$$

where cos is the trigonometric function called cosine.

Einstein, never an amiable loser, thought to change the direction of the conversation: "Vectors are members of a vector set, so for any two vectors **v1** and **v2**, another vector, **v1+v2,** is a member of the same space. What is **v1+v2**?"

Newton, agreeably, without noticing Einstein's tactic: "Let's also consider scalar multiplication."

Breton, taking up the new challenge: "Scalar multiplication is easy. If **v1** = q1∗**u**(**v1**) is a vector, then

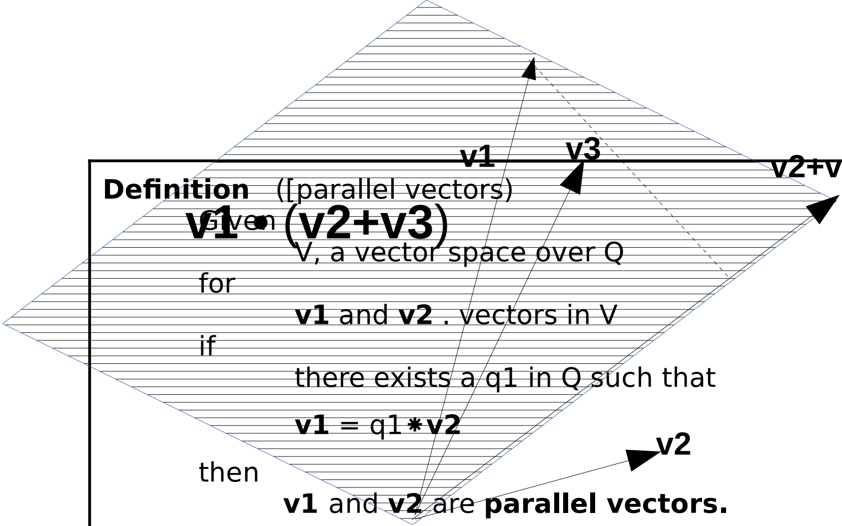
$$\mathbf{v2} \equiv q2 * \mathbf{v1} = q2 * q1 * \mathbf{u}(\mathbf{v1})$$

So scalar multiplication produces another vector having the same direction as the original vector, but with a scaled length. For this reason this kind of multiplication in the vector set is called scalar multiplication.

Two vectors having the same direction are called **parallel vectors**.

Newton, probing: "Suppose **v2** = (−1)∗**v1** are they still parallel?"

Breton: "Good remark, Newton. I need to be more precise. Let me offer the following definition.



Definition ([parallel vectors)

Given

V, a vector space over Q

for

v1 and **v2** . vectors in V

if

there exists a q1 in Q such that

v1 = q1∗**v2**

then

v1 and **v2** are **parallel vectors.**

Further if q1> 0
v1 and **v2** are **positively parallel vectors**;
and if q1< 0
v1 and **v2** are **negatively parallel vectors.**

end of definition

Newton, probing still: "How about **0**?”

Breton: "Arguing from the definition, the zero vector would be parallel to any vector in **V**.

Einstein, making a favorite point and attempting again to control the conversation: "So the zero vector is a special vector! Let's return to vector addition. How about **v1+v2**?”

Addition in the Set of Vectors

Breton, taking up the challenge gingerly: "The vector, **v1+v2**, will have a length and direction, so

v1+v2 = q∗**u**;

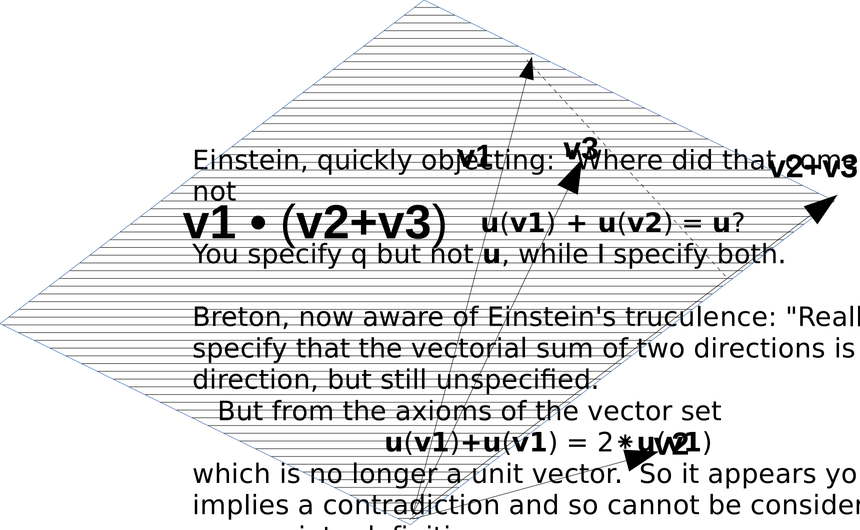
so to define vector addition we have to define q and **u**.

Einstein, with a touch of triumph in his voice: "How?”

Breton, pensively and cautiously: "Let's try with a simple example. Suppose **v1** and **v2** are directions. If so,

u(**v1**) **+ u**(**v2**) = q∗**u**
= 2 ∗cos(angle/2)∗**u**

where angle is the angle between **u**(**v1**) and **u**(**v2**).



Einstein, quickly objecting: "Where did that come from? Why not

u(**v1**) + **u**(**v2**) = **u**?

You specify q but not **u**, while I specify both.

Breton, now aware of Einstein's truculence: "Really? You specify that the vectorial sum of two directions is another direction, but still unspecified.

But from the axioms of the vector set

$$\mathbf{u}(\mathbf{v1})+\mathbf{u}(\mathbf{v1}) = 2*\mathbf{u}(\mathbf{v1})$$

which is no longer a unit vector. So it appears your suggestion implies a contradiction and so cannot be considered an appropriate definition.

Einstein, crushed, but too proud to concede: "Are there contradictions with your suggestion?

Breton: "Could be. As you pointed out, my suggestion does not specify the direction.

Thinking prudence called for, Einstein thought for a long moment on how best to challenge Breton. Finally, he mused: "I note that you have defined an angle from two directions. We might have defined direction in terms of angles. So which is a more fundamental concept: angle or direction?"

Newton, impulsively: "Angle!"

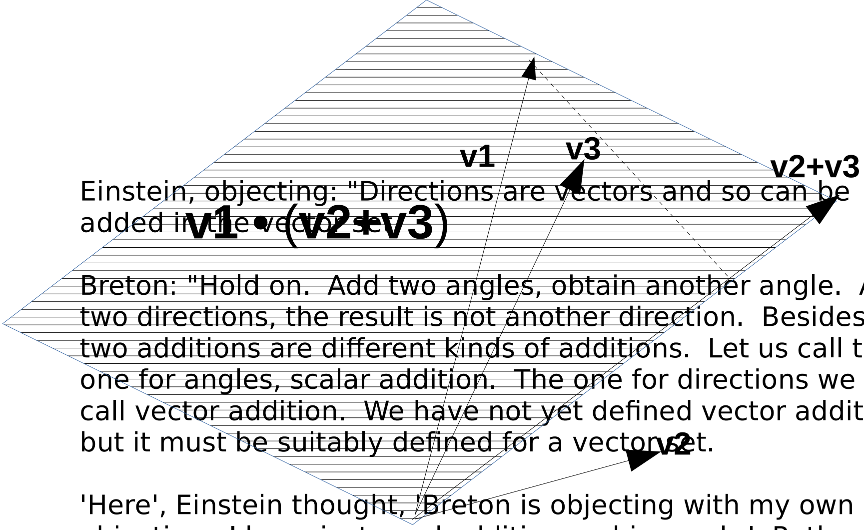
Einstein, glad to see his question taking root: "I say direction!"

Breton, always looking to reconcile his two friends: "Besides simple assertions, how can we come to the truth of the matter?"

Newton, resorting to a familiar tactic: "Let's enumerate the differences."

Breton: "Both words are used in many different contexts. Let us restrict our consideration to mathematical contexts which can lead to a physical application.

Angles can be added numerically. We can add a 90 degree angle to a 30 degree angle to form a 120 degree angle. Directions cannot be simply added to produce another direction."



Einstein, objecting: "Directions are vectors and so can be added in the vector set.

Breton: "Hold on. Add two angles, obtain another angle. Add two directions, the result is not another direction. Besides, the two additions are different kinds of additions. Let us call the one for angles, scalar addition. The one for directions we can call vector addition. We have not yet defined vector addition, but it must be suitably defined for a vector set.

'Here', Einstein thought, 'Breton is objecting with my own objection. I have just used addition ambiguously.' Rather than let Breton score that embarrassing point, Einstein quickly took up Newton's agenda: "Angles refer to triangles, whereas directions refer to a unit sphere.

Newton, continuing his agenda: "We can define either in terms of the other.

An angle can be defined from two directions originating from the same point, called the vertex. The two directions can then serve as sides of a triangle.

A direction can be defined in terms of angles. First set up a coordinate system of three mutually orthogonal axes. Using the axes as sides, a direction can be defined in terms of three angles.

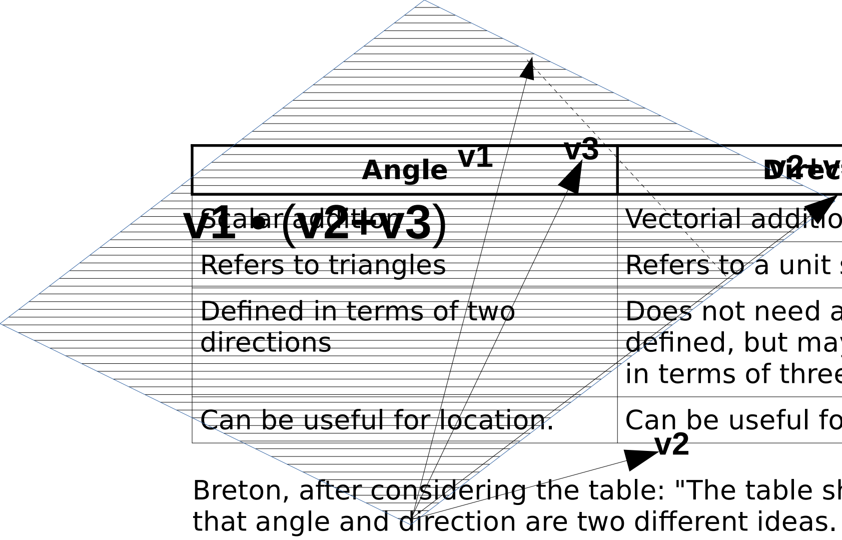
Breton: "True enough, but consider this, Newton. If Einstein asks me to point at you, I will simply point my index finger in your direction with no reference to angles at all.   So directions may be defined in terms of angles, but not necessarily so.

Newton: "Still locations can be defined in terms of angles, just as surveyors do. A baseline and two angles are all that is needed.

Einstein, enjoying the different points of view: "But locations are defined even more easily by a direction and a distance.

Breton: "Newton, would a table help us?”

With that Newton happily drew up the following table which he presented to his friends.



Angle	Direction
Scalar addition	Vectorial addition
Refers to triangles	Refers to a unit sphere
Defined in terms of two directions	Does not need angles to be defined, but may be defined in terms of three angles.
Can be useful for location.	Can be useful for location.

Breton, after considering the table: "The table shows clearly that angle and direction are two different ideas. For a mathematical science like Euclidean Geometry, angle may well be a better choice as prior to direction. The simplicity of direction seems more appropriate for Theoretical Physics.”

Turning to his two friends, he questioned:
”Do you both agree that angle and direction are ideas, not physical objects?

Newton: "Of course.”

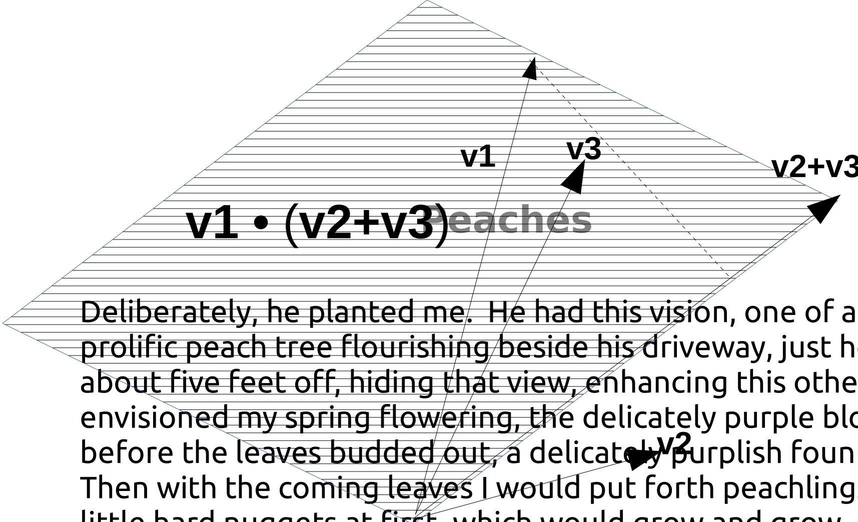
Einstein: "But they may be properties of material objects.

Breton: "Location appears a property of all material objects. That is why Theoretical Physics should favor direction as axiomatic in its vector set. So doing, locations are more simply described.

Will you accept then, for our great adventure, that direction is taken as a fundamental and axiomatic idea from which angles may be defined.

Newton quickly agreed, but Einstein commented: “It is a fine and subtle point which I accept reluctantly.”

The fineness and the subtlety put Breton in mind of a story about proper beginnings.



Peaches

Deliberately, he planted me. He had this vision, one of a prolific peach tree flourishing beside his driveway, just here about five feet off, hiding that view, enhancing this other. He envisioned my spring flowering, the delicately purple blooms before the leaves budded out, a delicately purplish fountain. Then with the coming leaves I would put forth peachlings, little hard nuggets at first, which would grow and grow. With the growing, my branches would begin to bow, almost to the ground.

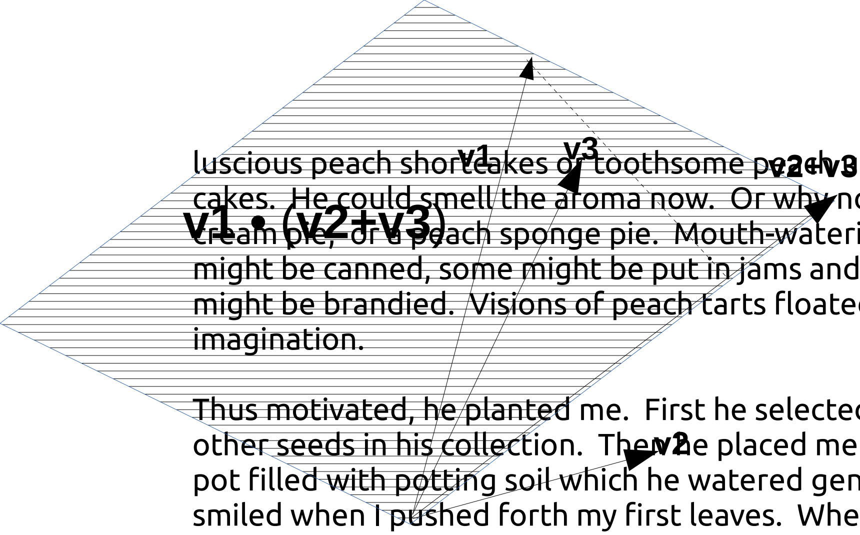
He imagined himself sitting in a chair by my trunk, lazily contemplating the the peachlings's slow growth. In my shade, he would read, or doze, or simply enjoy a comfortable peace.

The wrens would tell him when to harvest. Brashly, they would pick into the sunny side of the peach, a small indentation, leaving the firmer skin untouched. Time to harvest. A time for calling family and friends. A time for singing, for joyful collecting into baskets, bags, whatever, in which to collect the bountiful harvest. Peaches everywhere, in the kitchen, on the porch, in the fridge, on window sills.

And now the next steps would be launched. Peaches, washed and dried, could be served, whole, drilled or not, or peeled and with stones removed sliced just before serving. Cream could be added as a dessert, or they might find their way into a fruit cup or salad.

Or the peaches could be fried. He would cut the peaches in halves, remove the stones, and cook them over low heat until tender, basting with butter. He might relish the result as a meat accompaniment or even as a desert.

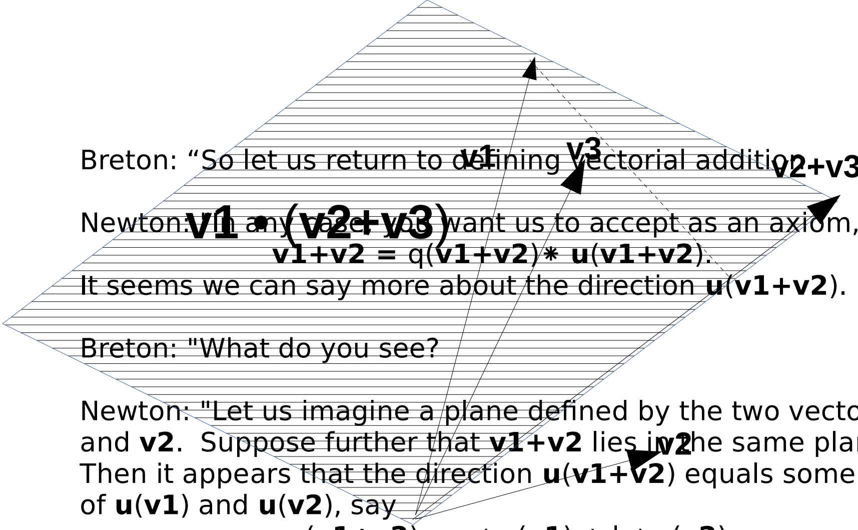
Or they might find their way into delicious peach cobblers, or



luscious peach shortcakes or toothsome peach upside down cakes. He could smell the aroma now. Or why not a peach cream pie, or a peach sponge pie. Mouth-watering. Some might be canned, some might be put in jams and jellies, some might be brandied. Visions of peach tarts floated by his imagination.

Thus motivated, he planted me. First he selected me from other seeds in his collection. Then he placed me in a five inch pot filled with potting soil which he watered generously. He smiled when I pushed forth my first leaves. When I grew to six inches, he transferred me from the pot to a large hole in just the location he had in mind. I grew fast. The first year I had grown two feet tall, the next year ten feet tall. He watered, he weeded, he mulched. The leaves, the bark looked exactly like a peach tree. Next year, he smiled to himself, he would be harvesting peaches.

Little does he know, I am an apricot.



Breton: “So let us return to defining vectorial addition.

Newton: "In any case you want us to accept as an axiom, that

v1+v2 = q(**v1+v2**)∗ **u**(**v1+v2**).

It seems we can say more about the direction **u**(**v1+v2**).

Breton: "What do you see?

Newton: "Let us imagine a plane defined by the two vectors **v1** and **v2**. Suppose further that **v1+v2** lies in the same plane. Then it appears that the direction **u**(**v1+v2**) equals some ratio of **u**(**v1**) and **u**(**v2**), say

u(**v1+v2**) = a∗**u**(**v1**) + b∗**u**(**v2**)

for some a and b.

Breton: "And the angle between **v1** and **v1+v2** or between **v2** and **v1+v2** must always be less that the angle between **v1** and **v2**.

Einstein, always looking to steer the conversation: " Let's do directions as a first step.

Breton: "OK. Suppose two directions, **u1** and **u2**. We know their sum as a vector of the vector set is not a direction. So

u1 + u2 = q∗**u3**

Further, **u1** and **u2** can be thought of a radii of a unit sphere. So if **u2** = **u1**, what might be an appropriate definition?

Newton, engagingly willlingly: "We should have

u1 + u1 = 2∗**u1**

Breton: "And how about **u1 +** (−**u1**)?

Newton: "We should have

u1 + (−**u1**) = **0**

Breton: "Now any direction **u2** will lie between **u1** and −**u1**, so their corresponding q's will lie between 2 and 0.

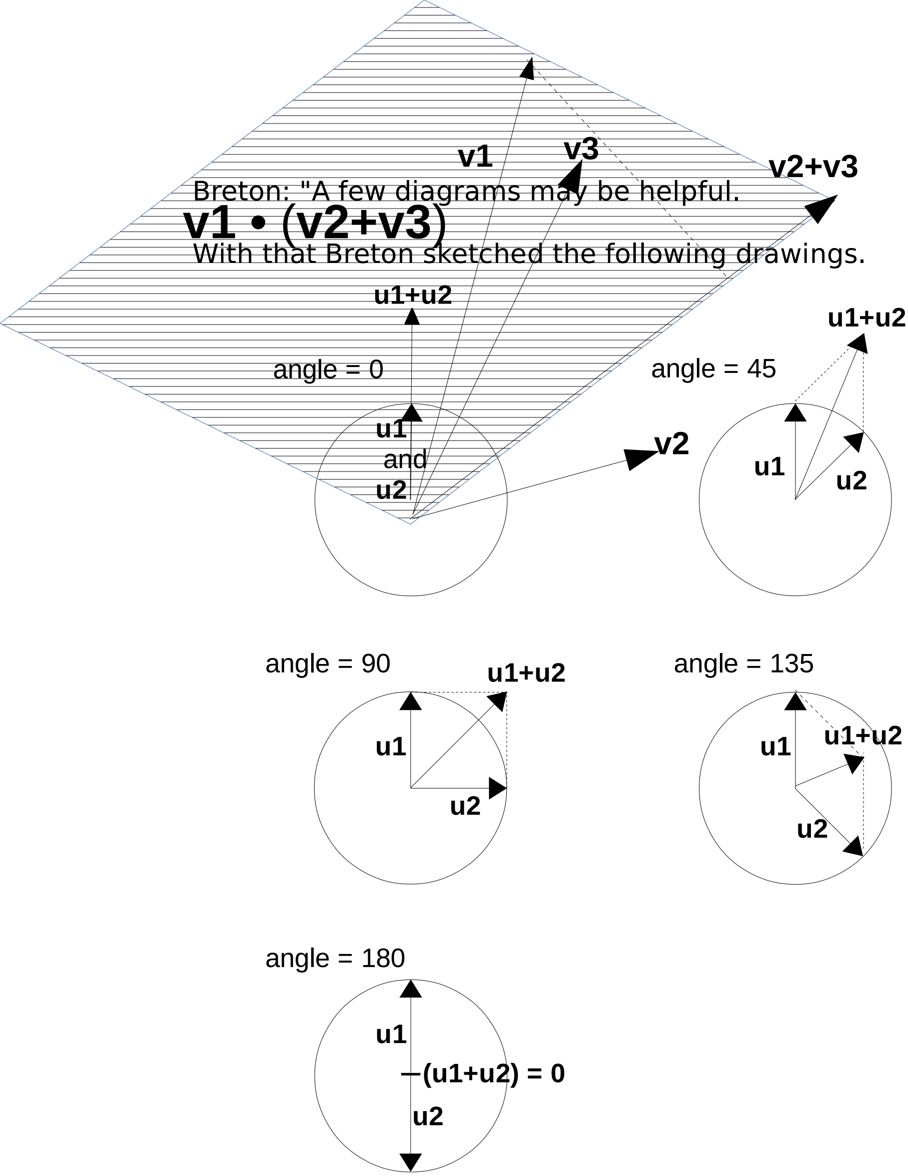
Newton: "And their direction?

Breton: "Half way way between them.

Einstein: "What does that mean?

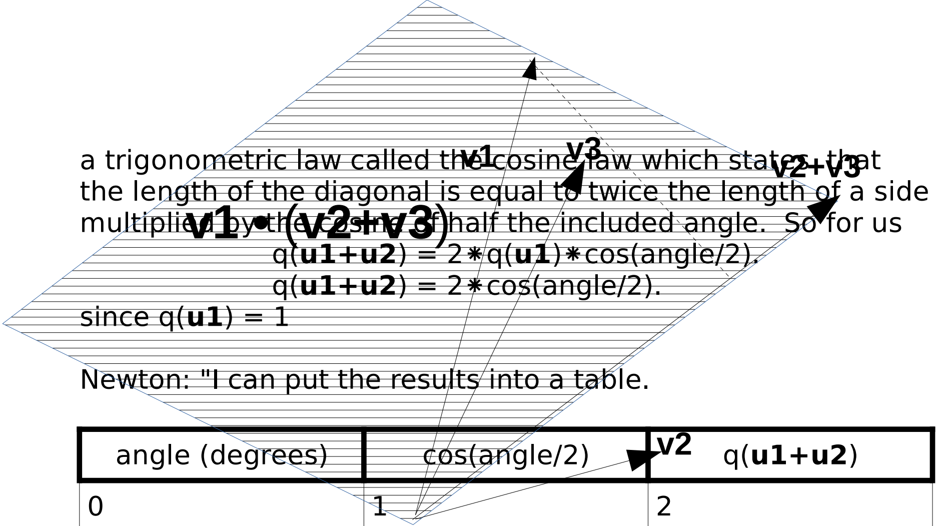
Breton: "A few diagrams may be helpful.

With that Breton sketched the following drawings.



Drawing *1:* **Addition of Directions**

Breton: "Look at the drawing carefully. The solid lines with arrows indicate vectors; the dashed lines are parallel to them. In each case a rhombus appears, that is a rectangle (quadrilateral) with four equal sides. The sum of the two directions is indicated by the diagonal of the rhombus. Half of the rhombus formed by **u1**, **u1+u2**, and the parallel **u2**, is a triangle. The sides of the triangle and its angles are related by



a trigonometric law called the cosine law which states that the length of the diagonal is equal to twice the length of a side multiplied by the cosine of half the included angle. So for us

q(**u1+u2**) = 2∗q(**u1**)∗cos(angle/2).

q(**u1+u2**) = 2∗cos(angle/2).

since q(**u1**) = 1

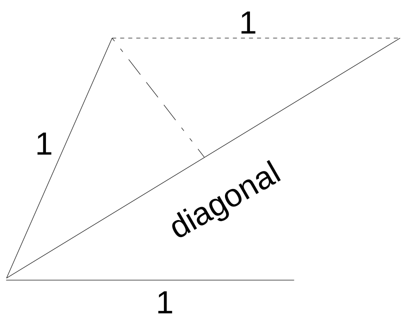
Newton: "I can put the results into a table.

angle (degrees)	cos(angle/2)	q(**u1+u2**)
0	1	2
45	0.92388	1.846776
90	sqrt(2)/2	1.41422
135	0.38638	0.77276
180	0	0

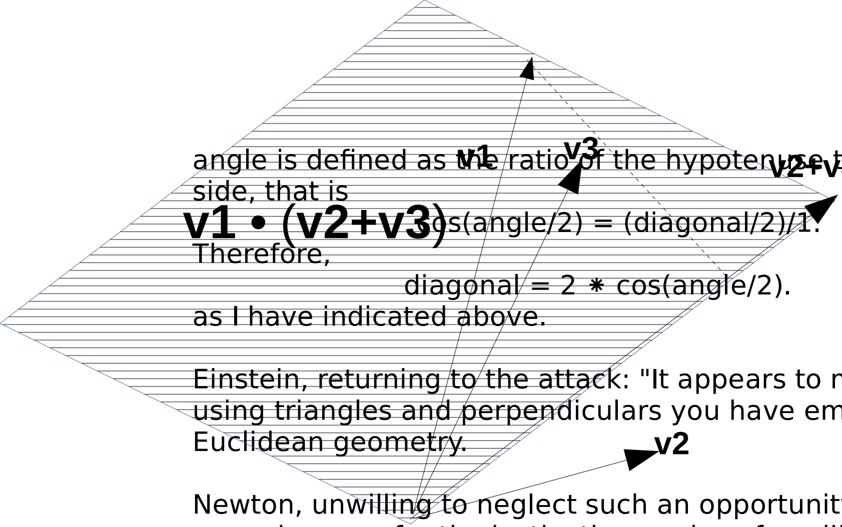
Einstein: "Justify your formula for q(**u1+u2**)!

Breton: "This is simply an exercise in trigonometry. Follow along in this diagram.

With that Breton handed his two friends the following diagram.



Breton: "First from the tip of the first direction, drop a line perpendicular to the diagonal line, as shown. Then note that the perpendicular line divides the larger triangle into two other equal right triangles. Moreover, the angle between the first direction and the diagonal is just half the angle between the vectors. Now in such a configuration the cosine of an



angle is defined as the ratio of the hypotenuse to the adjacent side, that is

cos(angle/2) = (diagonal/2)/1.

Therefore,

diagonal = 2 * cos(angle/2).

as I have indicated above.

Einstein, returning to the attack: "It appears to me that by using triangles and perpendiculars you have embraced Euclidean geometry.

Newton, unwilling to neglect such an opportunity: "And given a good reason for the instinctive genius of my illustrious forebear in basing his Physics on Euclidean geometry.

Breton, retreating: "Einstein is right. By insisting on measuring the diagonal, I have lost the path. Theoretical Physics should not be tied to Euclid's geometry, or indeed to any geometry at all. Nor should our vector set. I have made specific what might well have been left unspecified. Still the process of imagining the sum of two vectors from the image of a rhomboid can stand, provided we do not tie the rhomboid to a Euclidean plane.

Newton: "You are denigrating my illustrious ancestor.

Breton: "Not only yours, but Einstein's as well.

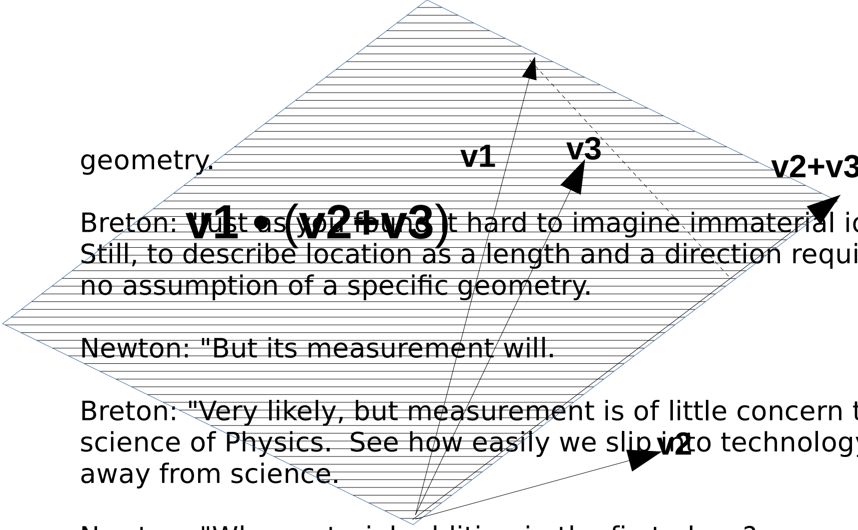
Einstein, still prodding: "Some sort of geometry *has* to be assumed for Physics.

Breton: "And if it doesn't correspond with reality?

Unable to respond both Newton and Einstein fell silent.

Breton: "We are engaged in a effort to create ideas which correspond to physical reality. To geometrize the description of location may impose an assumption which leads physicists astray. Theoretical Physics needs only conceive of a vector set with vectorial addition satisfying the axioms. Addition in the vectorial set is merely *illustrated* by the diagonal of the rhomboid.

Einstein: "I find it difficult to think about location without a



geometry.

Breton: "Just as you found it hard to imagine immaterial ideas. Still, to describe location as a length and a direction requires no assumption of a specific geometry.

Newton: "But its measurement will.

Breton: "Very likely, but measurement is of little concern to the science of Physics. See how easily we slip into technology and away from science.

Newton: "Why vectorial addition in the first place?

Breton: "We observe physical objects as mutable. An object lextended in one direction may subsequently be extended in a different direction; an object moving in a certain direction, may subsequently be observed moving in another direction. An object being forced in one direction may subsequently be forced in another direction. A mathematical vector set has the possibility of being transformed into an appropriate concept for Theoretical Physics to describe and understand these observations.

Einstein: "How do you finally describe vectorial addition for any two vectors in our special mathematical vector set?

Breton: "First let me review what we have learned from directions. The specific definition of addition for *directions* fits some of the axioms of a vector set. Let me list them.

$$\mathbf{u1} + \mathbf{u2} = \mathbf{u2} + \mathbf{u1}$$

$$1 * (\mathbf{u1} + \mathbf{u2}) = 1 * \mathbf{u1} + 1 * \mathbf{u2}$$

$$(1+1) * \mathbf{u1} = 1 * \mathbf{u1} + 1 * \mathbf{u1}$$

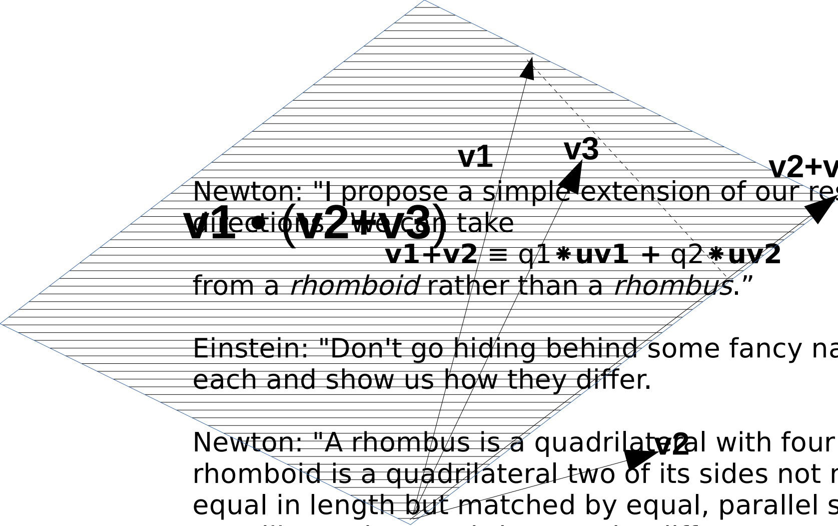
$1 * \mathbf{u} = \mathbf{u}$, for any direction

and for every direction $\mathbf{u}$, there exists a vector $-\mathbf{u}$ such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

Einstein: "But not all the axioms are satisfied.

Breton: "True enough. Remember we started the investigation of the plus operator for *vectors* by first considering what might be appropriate for *directions*. Now we can climb a little higher to consider addition for vectors generally.



Newton: "I propose a simple extension of our results for directions. We can take

v1+v2 ≡ q1∗**uv1** + q2∗**uv2**

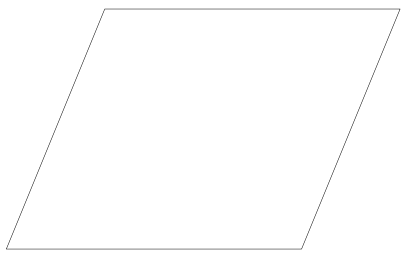
from a *rhomboid* rather than a *rhombus*."

Einstein: "Don't go hiding behind some fancy names. Explain each and show us how they differ.

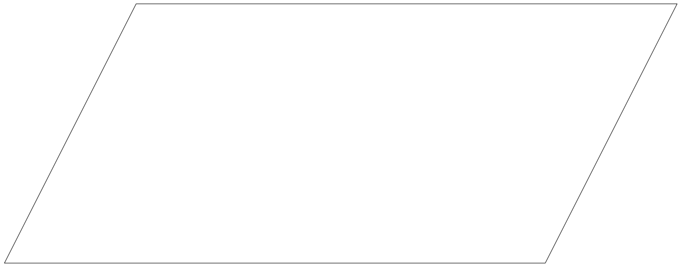
Newton: "A rhombus is a quadrilateral with four equal sides. A rhomboid is a quadrilateral two of its sides not necessarily equal in length but matched by equal, parallel sides.

An illustration can bring out the difference perhaps more clearly than words. Breton, would you kindly draw us a rhombus and a rhomboid.

Breton quickly obliged with the following drawings.

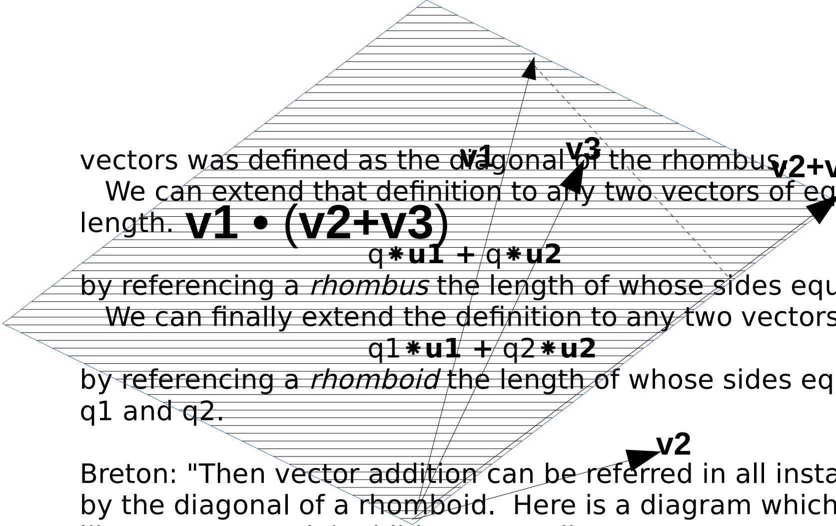


rhombus



rhomboid

Newton: "For directions we used a rhombus each of whose sides had a length equaled one. Then the addition of two



vectors was defined as the diagonal of the rhombus.
We can extend that definition to any two vectors of equal length.

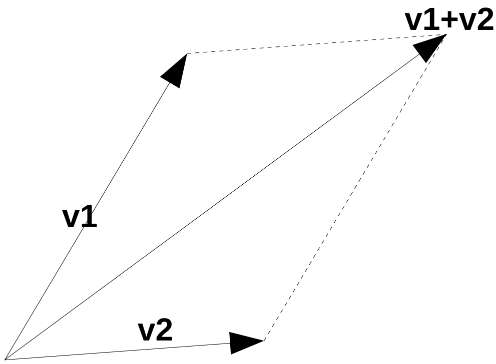
q**∗u1 +** q**∗u2**

by referencing a *rhombus* the length of whose sides equals q.
We can finally extend the definition to any two vectors

q1**∗u1 +** q2**∗u2**

by referencing a *rhomboid* the length of whose sides equals q1 and q2.

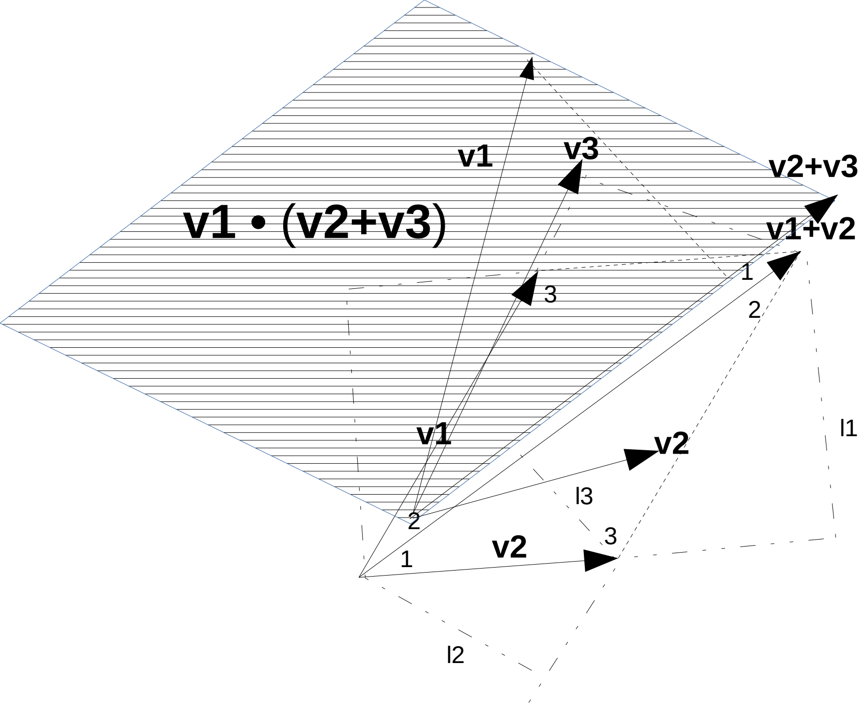
Breton: "Then vector addition can be referred in all instances by the diagonal of a rhomboid. Here is a diagram which illustrates vectorial addition generally.



vector addition

Einstein: "The drawing shows what you are trying to define, but what is the length of **v1+v2**?

Breton: "Here this sketch may help you.



Breton: "This sketch labels the three angles: 1,2, and 3, and includes extension lines so the their sines can be indicated. Angle1 lies opposite **v1**; angle2 lies opposite **v2**; angle3 lies opposite **v1+v2**. Since

sin(angle1) = length(l1)/length(**v1+v2**)

length(**v1+v2**) = length(l1)/sin(angle1)

Since sin(angle2) = length(l2)/length(**v1+v2**)

length(**v1+v2**) = length(l2)/sin(angle2)

Both angle1 and angle2 are acute angles, but angle3 is obtuse. Referring to angle3 the length of the sum can be expressed in terms of cosines.

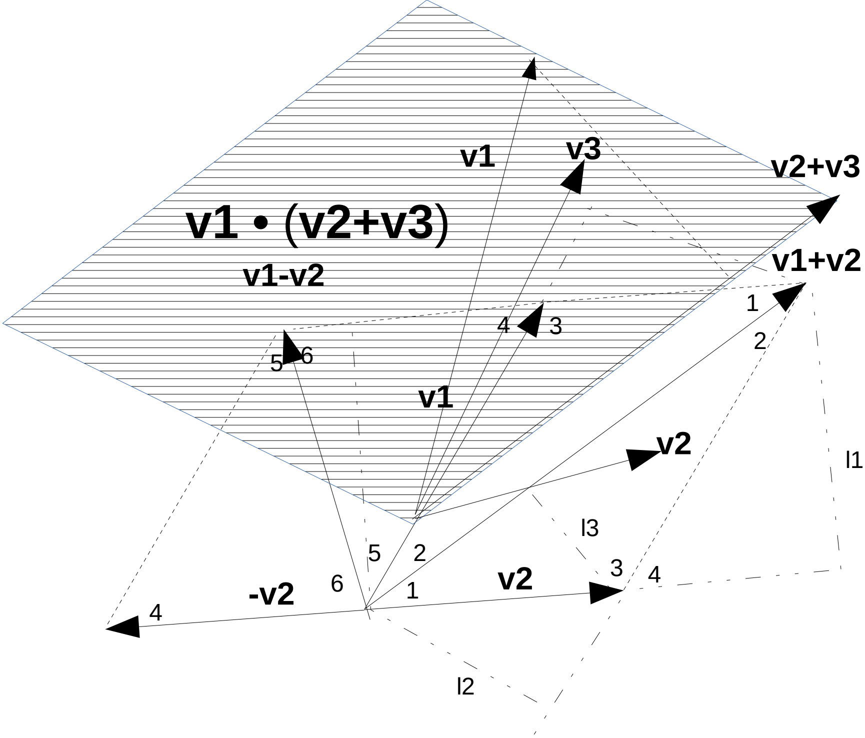
length(**v1+v2**) = cos(angle1)/length(**v2**)
+ cos(angle2)/length(**v1**)

So here Einstein are three equations for length(**v1+v2**).

Einstein, continuing the challenge: "Express the difference between two vectors!

Breton: "I will have to expand my sketch a little.

With that Breton quickly handed his friends the following sketch.



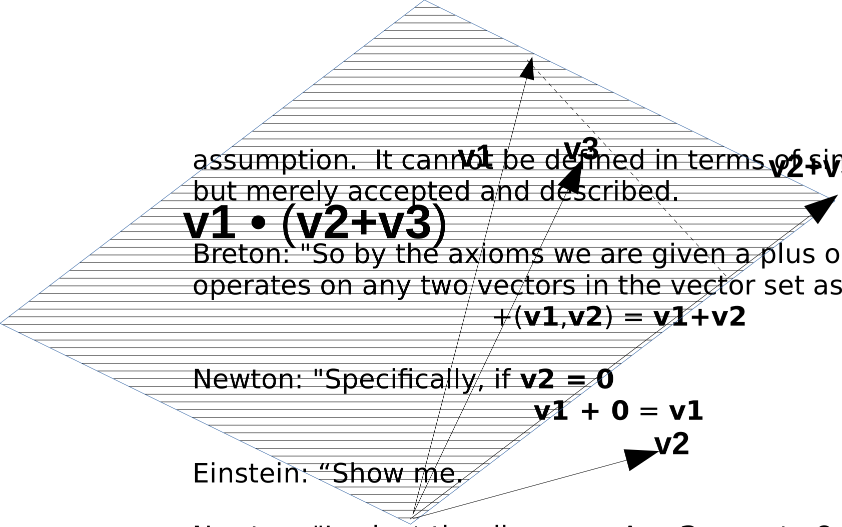
Breton: "Now three other angles have been indicated: angle4, angle5, and angle 6. From the sketch we see
sin(angle4)/length(**v1−v2**)
= sin(angle5)/length(**−v2**)
= sin(angle6)/length(**v1**)
so that
length(**v1−v2**) = length(**−v2**)∗sin(angle4)/sin(angle5)
length(**v1−v2**) = length(**v1**)∗sin(angle4)/sin(angle6)

The answer may involve us again in a specific geometry and lead us off our chosen path. For our purposes we will simply accept as axiomatic that our vector set has an addition operator which operates on any two vectors as referenced in our rhomboid illustration without implying a specific geometry.

Einstein: "So what can you give for a *definition* of vectorial addition.

Breton: "Nothing. Addition in the vector set is an axiomatic assumption. It can be described, but not defined since definition would imply something 'more' axiomatic.

Newton: "In Euclidean Geometry, a 'line' is an axiomatic



assumption. It cannot be defined in terms of simpler axioms, but merely accepted and described.

Breton: "So by the axioms we are given a plus operator which operates on any two vectors in the vector set as

+(**v1**,**v2**) = **v1+v2**

Newton: "Specifically, if **v2 = 0**

v1 + 0 = **v1**

Einstein: “Show me.

Newton: "Look at the diagram. As **v2** goes to **0**, angle 2 also becomes zero and **v1+v2** becomes **v1**.

Einstein: “And if **v2** = **−v1**?

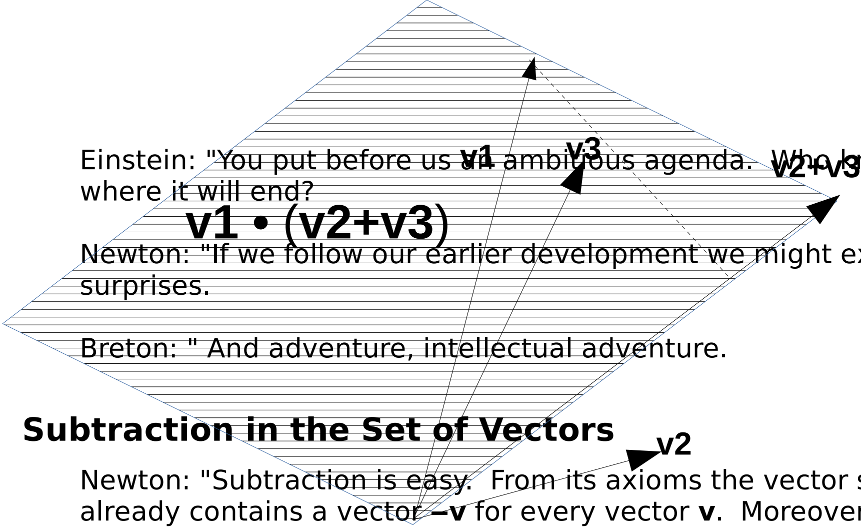
Newton: "Look at the diagram again. Let **v2** become **−v1.** Then angle 2 plus angle 1 equal pi and **v1+v2** becomes **0** orthogonal to **v1**.
...So we see the rhomboid scheme leads to a definition of the **+** vectorial operator consistent with the axioms for a the vectorial set.
The plus operator acts symbolically like the plus operator for integers.

Breton: "So let us accept that a set symbolized as
V ={{q∗**u** such that q is an element of Q, **u** a direction},+}
as a vector set. since it satisfies all the axioms of a mathematical vector set.

Newton: "I agree.

Einstein: "I also agree, but where is all this leading to?

Breton: "Remember how we developed the quotient numbers, starting with the positive integers? We moved from the positive integers, to the negative integers, to multiplication, to division, each time enlarging our consideration to a set which finally supported a full algebra. Then we showed that the set of quotient partitions could support a topology from which we could define limits and then continuous functions. Do you think our vector set could support a similar development?



Einstein: "You put before us an ambitious agenda. Who knows where it will end?

Newton: "If we follow our earlier development we might expect surprises.

Breton: " And adventure, intellectual adventure.

Subtraction in the Set of Vectors

Newton: "Subtraction is easy. From its axioms the vector set already contains a vector **−v** for every vector **v**. Moreover we accept that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

as we have seen in the illustration for addition in the vector set.

Breton: "Can minus act like an operator?

Newton: "The axioms give us plus as an operator, but not minus. If minus were an operator we would need to know

$$\mathbf{v1} - \mathbf{v2}$$

for any **v1** and **v2**.

Breton: "We already know

$$\mathbf{0} + (-\mathbf{v2}) = -\mathbf{v2}$$

which we could take as

$$\mathbf{0} - \mathbf{v2} = -\mathbf{v2}$$

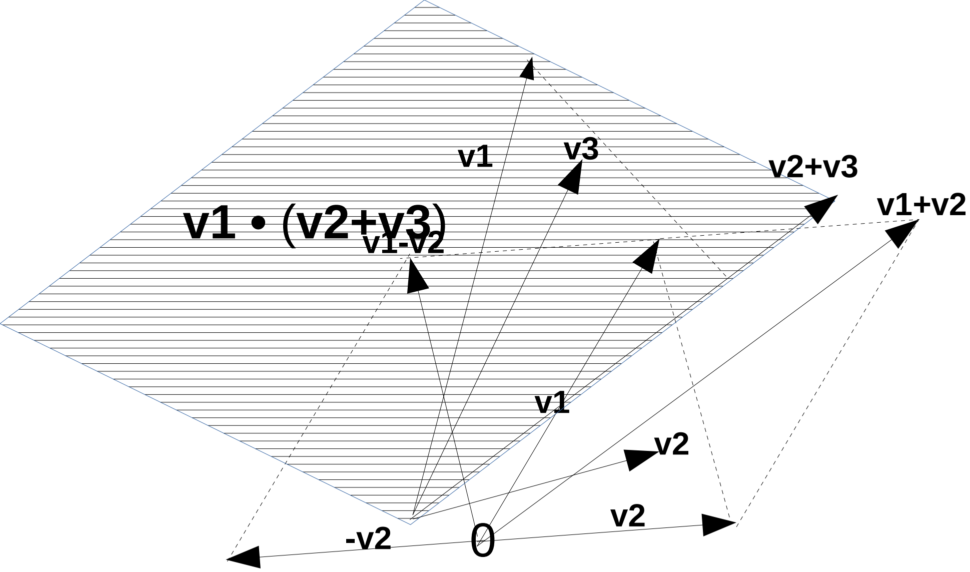
Newton: "Minus would be well defined as an operator as

$$\mathbf{v1} - \mathbf{v2} \equiv \mathbf{v1} + (-\mathbf{v2})$$

Einstein: "Breton, show us how this would look as an illustration.

Breton: "Gladly.

With that he quickly produced the following drawing.



vector addition and subtraction

Breton: "Using the same rules of vectorial addition, you can see that **v1−v2** may differ from **v1+v2** in both length and direction.

Newton: "Notice the line parallel to **v1−v2**, the one stretching from the tip of **v2** to the tip of **v1**. It has the same length and direction as **v1−v2.**

Einstein: "Same length, but not a direction since it does not relate to the unit sphere.

Breton: "It would if the unit sphere were centered at the tip of **v2** instead of **0**.

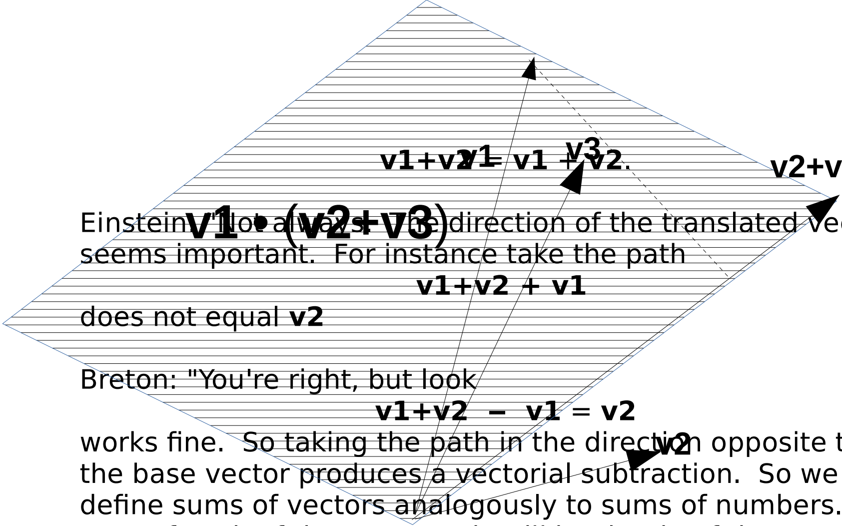
Newton: "Look at the dashed line parallel to **v2**. We would arrive at **v1+v2** by traveling along **v1** and the parallel line.

Breton: "And we would arrive at **v1−v2** by traveling along **v1** and the line parallel to **−v2**. Have you discovered a new way for defining addition and subtraction in the vector set?

Newton: "Yes we have. I am tracing some paths in the diagram. They all comply with the rule.

Breton: "So the parallel lines can be thought of as translated base vectors. Allowing translated vectors enables vectors to be added and subtracted. For instance,

v1+v2 = v1 + v2.



Einstein: "Not always. The direction of the translated vector seems important. For instance take the path

v1+v2 + v1

does not equal **v2**

Breton: "You're right, but look

v1+v2 − v1 = **v2**

works fine. So taking the path in the direction opposite than the base vector produces a vectorial subtraction. So we can define sums of vectors analogously to sums of numbers. The start of each of the summands will be the tip of the arrow of its previous member. The summand will be plus or minus depending on its correspondence to its base vector.

Newton: "More than analogous. If we take the partitions of Q as a vector set with only two directions, plus and minus, then the analogy becomes perfect. So we have achieved a generalization of numerical Arithmetic for our vector set.

Breton: "Another of your splendid insights. But our notation has not followed this new way of vectorial Arithmetic. Let me propose a similar expansion of our notation. Our vectors have been designated as

v = q(**v**)∗**u**(**v**).

Each translated vector can be written as

v = **v+v0 − v0**

one for each **v0**. That is,

q(**v**)∗**u**(**v**) = q(**v+v0**)∗**u**(**v+v0**) − q(**v0**)∗**u**(**v0**)

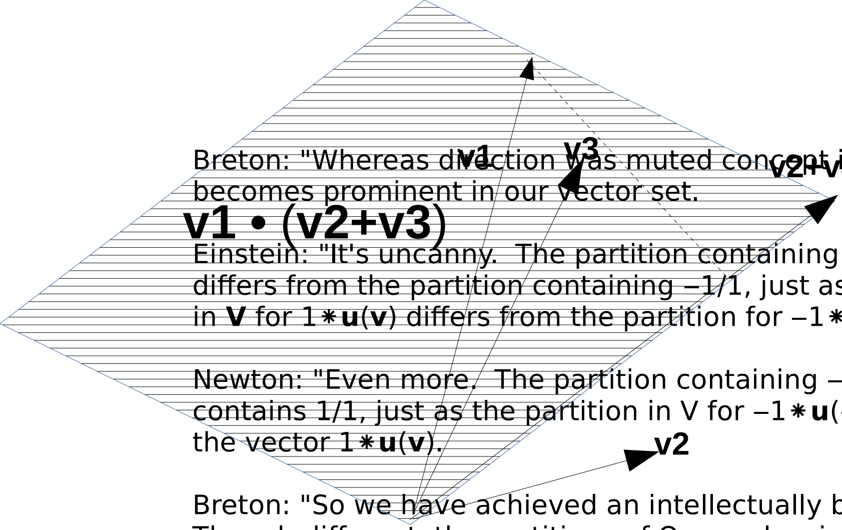
So for a translated vector we use

v = **v2 – v0**

where **v2** = **v+v0**. A translated vector of **v** can be thought of as starting from a base vector **v0** and extending in the direction **u**(**v**) for a length q(**v**) to the vector **v2**. A translated vector can be found for each base vector **v0**.

Einstein: "Then our former notation can be seen as having implied **v0** = **0** for the base vector.

Newton: "Something like the partitions of Q. A given vector and all its translations acts like a partition in the vector set, one for each value of q and each direction.



Breton: "Whereas direction was muted concept in Q, it becomes prominent in our vector set.

Einstein: "It's uncanny. The partition containing 1/1 in Q differs from the partition containing −1/1, just as the partition in **V** for $1*\mathbf{u}(\mathbf{v})$ differs from the partition for $-1*\mathbf{u}(\mathbf{v})$.

Newton: "Even more. The partition containing −1/(−1) contains 1/1, just as the partition in V for $-1*\mathbf{u}(-\mathbf{v})$ contains the vector $1*\mathbf{u}(\mathbf{v})$.

Breton: "So we have achieved an intellectually beautiful vista. Though different, the partitions of Q reveal a similarity to the partitions of **V**. The perception of an underlying unity gives us a better appreciation of both and brings us intellectual enjoyment.

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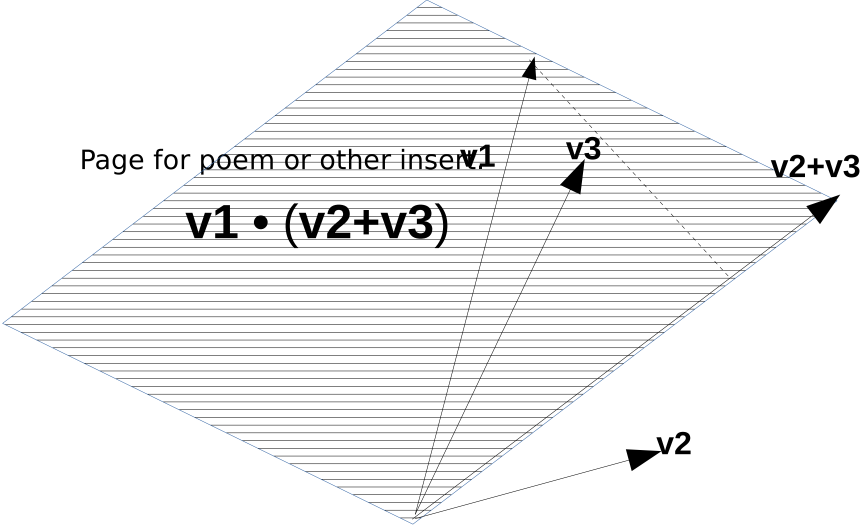
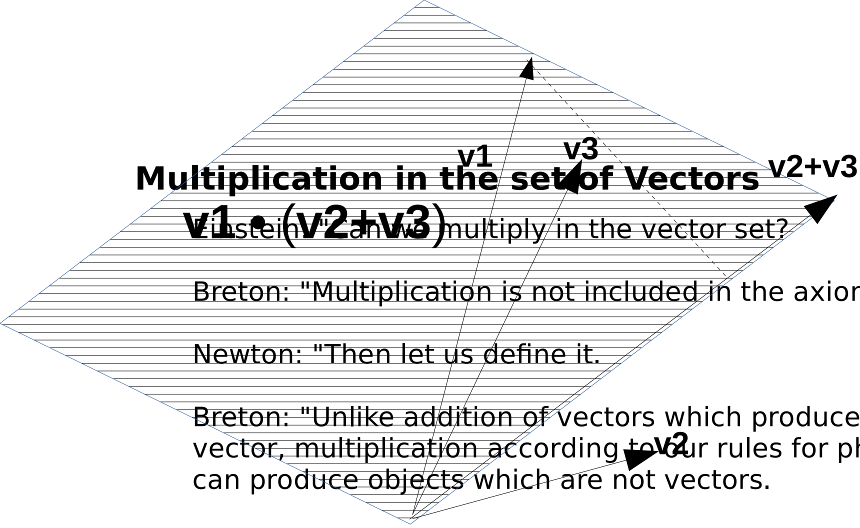


Multiplication in the set of Vectors



Einstein: "Can we multiply in the vector set?

Breton: "Multiplication is not included in the axioms.

Newton: "Then let us define it.

Breton: "Unlike addition of vectors which produces another vector, multiplication according to our rules for physical units can produce objects which are not vectors.

Einstein: "Why be restricted to rules for labels when we are defining a mathematical structure?

Breton: "We are looking to define mathematical objects which can be transformed into Theoretical Physics. So it makes sense to respect the restrictions even in mathematics. Remember how we just used the same principle when we refused to add scalar variables with vector variables.

Newton, impatiently: "Agreed. So how do we proceed?

Breton, plodding forward: "Let's enumerate the possibilities. The product resulting from the multiplication of two vectors. could be a member of the underlying field, Q. Or again, it could be another vector.

Einstein: "Then these would be two different kinds of multiplication.

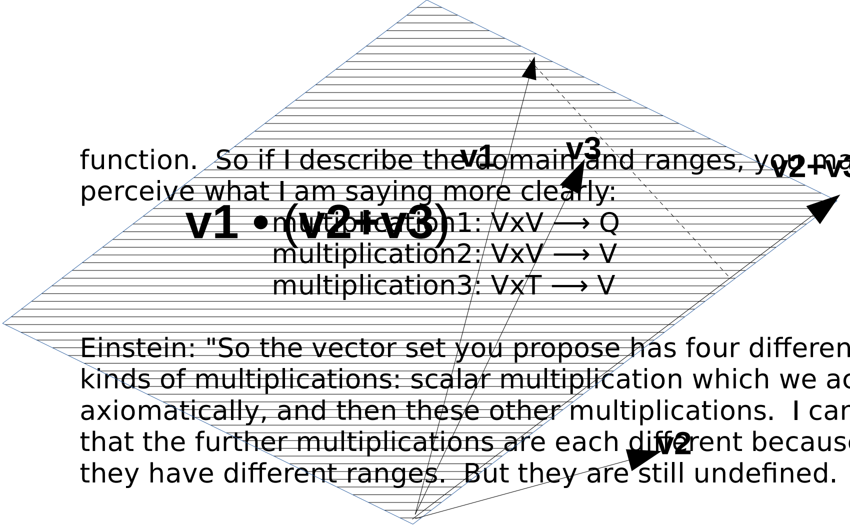
Breton: "Correct. And let me add still another product, a transformation.

Newton: "What kind of transformation?

Breton: "The transformation would take one vector and transform it into another vector. Although involving vectors, the transformation itself is not a vector.

Einstein: "So far I hear words; please show us concretely what you mean?

Breton, patiently: "Will this help? Multiplication is a kind of



function. So if I describe the domain and ranges, you may perceive what I am saying more clearly:

multiplication1: VxV ⟶ Q
multiplication2: VxV ⟶ V
multiplication3: VxT ⟶ V

Einstein: "So the vector set you propose has four different kinds of multiplications: scalar multiplication which we accept axiomatically, and then these other multiplications. I can see that the further multiplications are each different because they have different ranges. But they are still undefined.

Inner (Dot) Product

Breton: "So let us start with multiplication1

Definition (inner (dot) product)
Given
v1, **v2** vectors in the vector space.
for
v1 = q(**v1**)∗**u**(**v1**)
v2 = q(**v2**)∗**u**(**v2**)
angle, the angle between **u**(**v1**) and **u**(**v2**)
then

v1 • v2 ≡ q(**v1**)∗q(**v2**)∗cos(angle)

end of definition

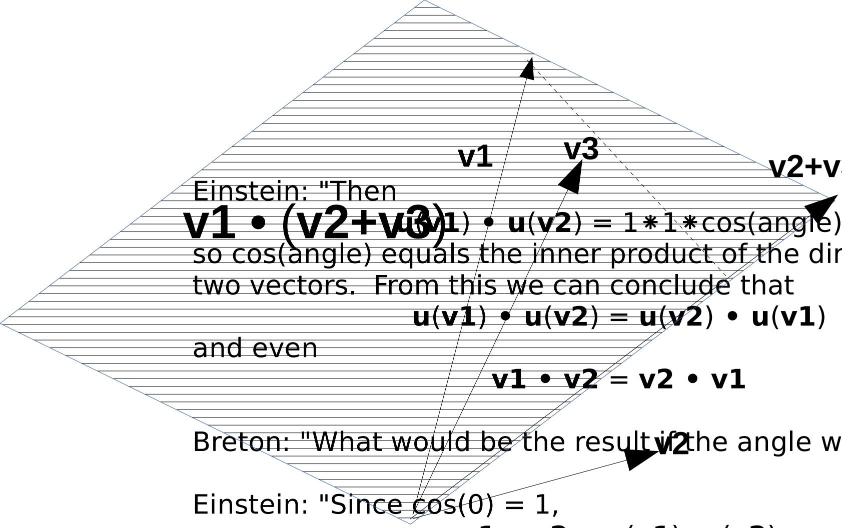
Multiplication1 is called the **inner product** or sometimes the **dot product**. By this curious convention, we call the function, •, by its image. To avoid confusion with the other multiplications in the vector se, it is symbolized with **'•'**.
As you can see

•: VxV ⟶ Q

where VxV is a joint set.

Einstein, analytically: "The product depends on the angle between the two vectors.

Breton: "Correct. Suppose both vectors are unit vectors. What would be the result?



Einstein: "Then

$$\mathbf{u}(\mathbf{v1}) \bullet \mathbf{u}(\mathbf{v2}) = 1 * 1 * \cos(\text{angle})$$

so cos(angle) equals the inner product of the directions of the two vectors. From this we can conclude that

$$\mathbf{u}(\mathbf{v1}) \bullet \mathbf{u}(\mathbf{v2}) = \mathbf{u}(\mathbf{v2}) \bullet \mathbf{u}(\mathbf{v1})$$

and even

$$\mathbf{v1} \bullet \mathbf{v2} = \mathbf{v2} \bullet \mathbf{v1}$$

Breton: "What would be the result if the angle were 0?

Einstein: "Since cos(0) = 1,

$$\mathbf{v1} \bullet \mathbf{v2} = q(\mathbf{v1}) * q(\mathbf{v2})$$

Breton: "And if **v1** = **v2**?

Einstein: "Then

$$\mathbf{v1} \bullet \mathbf{v1} = q(\mathbf{v1}) * q(\mathbf{v1})$$

Breton: "So then the inner product of a vector with itself is equal to the square of its length.”

Einstein: "Interesting.

Breton: "Suppose angle equals 90 degrees.

Einstein: "Then cos(angle) = 0, so

$$\mathbf{v1} \bullet \mathbf{v2} = 0$$

Breton: "Two vectors so related are said to be **perpendicular** to each other, also called **orthogonal** vectors.
And if the angle equals 180 degrees?

Einstein: "Then cos(angle) = –1, so

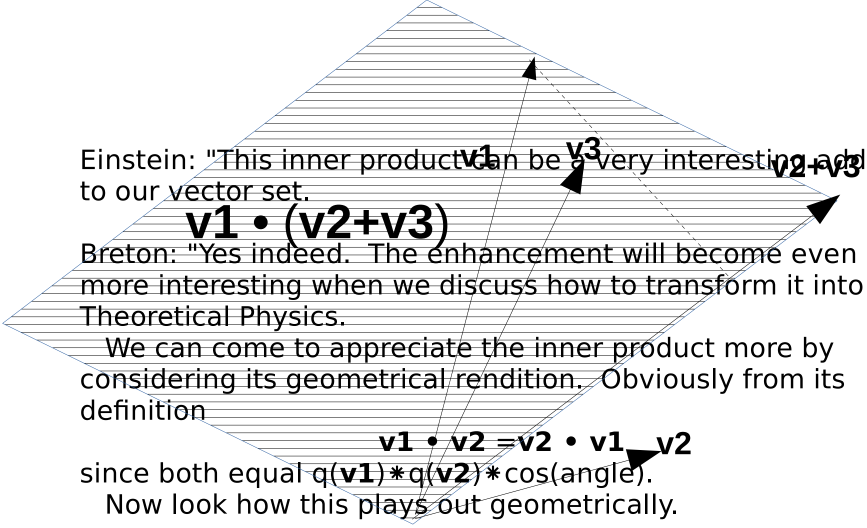
$$\mathbf{v1} \bullet (-\mathbf{v2}) = -q(\mathbf{v1}) * q(\mathbf{v1})$$

Breton: "Which would be the same as

$$(-\mathbf{v1}) \bullet \mathbf{v2} = -q(\mathbf{v1}) * q(\mathbf{v2})$$

Einstein: "correct.

Breton: "So the inner product varies from q(**v1**)∗q(**v2**) to –q(**v1**)∗q(**v2**) depending on the alignment of the two vectors.



Einstein: "This inner product can be a very interesting addition to our vector set.

Breton: "Yes indeed. The enhancement will become even more interesting when we discuss how to transform it into Theoretical Physics.

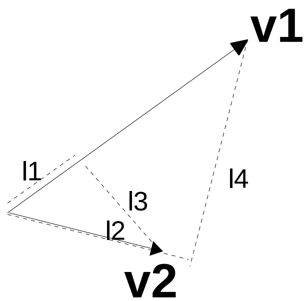
We can come to appreciate the inner product more by considering its geometrical rendition. Obviously from its definition

v1 • v2 =**v2 • v1**

since both equal q(**v1**)∗q(**v2**)∗cos(angle).

Now look how this plays out geometrically.

With that Breton handed the following sketch to his friends.



Inner product

Breton: "In addition to the two vectors, the sketch shows two right triangles composed of the lines: l2, l4, q(**v1**) and l1, l3, q(**v2**). From the sketch

cos(angle) = l2/ q(**v1**) = l1/ q(**v2**)

Now consider

v1 • v2 = q(**v1**)∗q(**v2**)∗cos(angle)
= q(**v1**)∗q(**v2**)∗l2/ q(**v1**)

Similarly

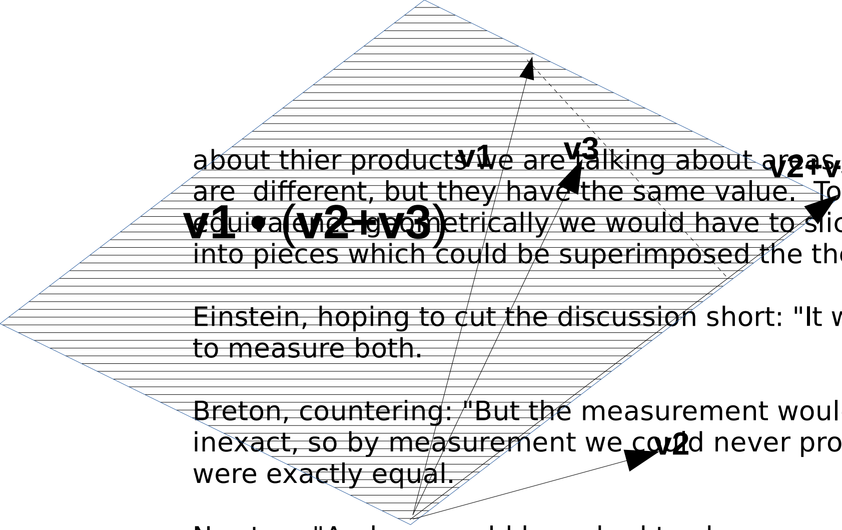
v1 • v2 = q(**v1**)∗q(**v2**)∗l1/ q(**v2**)

So

q(**v2**)∗l2 = q(**v1**)∗l1

a result somewhat difficult to see geometrically. Thus we can often use a result easily proved vectorially to establish a result much more difficult to prove geometrically. And vice−versa.

Newton: "Since both q(**v**)'s and l's are lengths, when we talk



about thier products we are talking about areas. The areas are different, but they have the same value. To prove their equivalence geometrically we would have to slice up one area into pieces which could be superimposed the the second area.

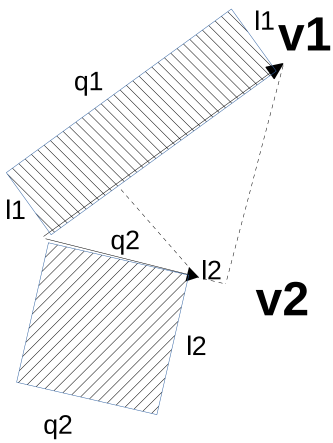
Einstein, hoping to cut the discussion short: "It would be easier to measure both.

Breton, countering: "But the measurement would always be inexact, so by measurement we could never prove the areas were exactly equal.

Newton: "And we would have had to choose some unit of measurement.

Einstein: "Breton, show us a sketch of the area of an inner product.

Complying Breton produced the following sketch.



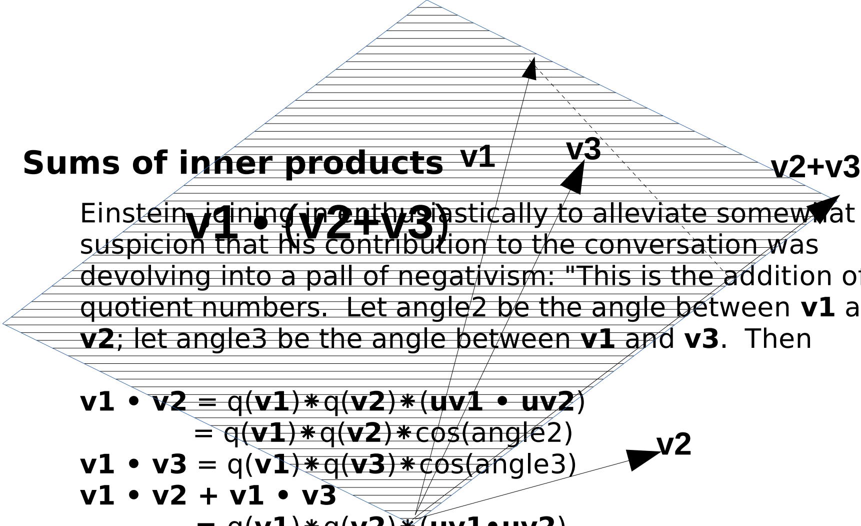
Breton: "Both of the hatched areas equal **v1 • v2**.

Breton,with a note of urgency: "So we come to appreciate the beauty and harmony of the inner product. But let's move on.
We have axiomatically that

v1 + (v2 + v3) = (v1 + v2) + v3

What can we say about (**v1 • v2**) + (**v1 • v3**)?

Sums of inner products



Einstein, joining in enthusiastically to alleviate somewhat a suspicion that his contribution to the conversation was devolving into a pall of negativism: "This is the addition of two quotient numbers. Let angle2 be the angle between **v1** and **v2**; let angle3 be the angle between **v1** and **v3**. Then

v1 • v2 = q(**v1**)∗q(**v2**)∗(**uv1 • uv2**)
= q(**v1**)∗q(**v2**)∗cos(angle2)
v1 • v3 = q(**v1**)∗q(**v3**)∗cos(angle3)
v1 • v2 + v1 • v3
= q(**v1**)∗q(**v2**)∗(**uv1•uv2**)
+ q(**v1**)∗q(**v3**)∗(**uv1•uv3**)
= q(**v1**)∗(q(**v2**)∗(**uv1•uv2**) +q(**v3**)∗(**uv1•uv3**)
= q**(v1**)∗(q(**v2**)∗cos(angle2) + q(**v3**)∗cos(angle3))

Breton appreciatively : "What can we say about
v1 • (**v2+v3**)?

Einstein: "That's not hard. Let
v2 = q(**v2**)**∗u**(**v2**)
and **v3** = q(**v3**)**∗u**(**v3**).
Then
v1 • (**v2+v3**)
= q(**v1**)**∗u**(**v1**)•(q(**v2+v3**)**∗u**(**v2+v3**)

Breton: "So does **v1 •** (**v2+v3**) = **v1 • v2 + v1 • v3**?

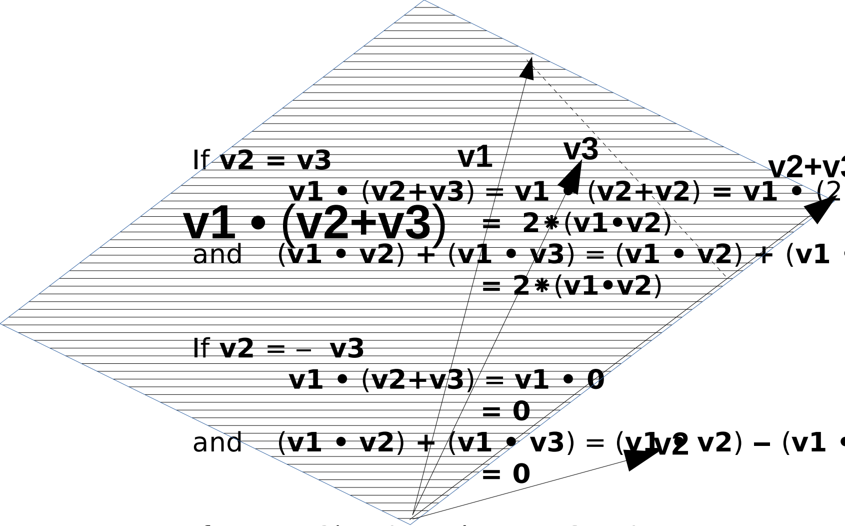
Einstein: "The formulas are almost the same

Breton: "But not exactly. The would be equal if
(q(**v2**)∗(**uv1•uv2**) + q(**v3**)∗(**uv1•uv3**)
= u(**v1**)•(q(**v2**)**∗u**(**v2**)**+**q(**v3**)**∗u**(**v3**))

Einstein, recalling the earlier discussion on inner products: "Could they possible be equal if not exactly the same?

Breton: "An intriguing possibility. Le us take same examples.

If **v2** or **v3** = 0, say **v2** = 0, then
v1 • (**v2+v3**) = **v1 • v3**
and (**v1 • v2**) **+** (**v1 • v3**) = **v1 • v3**
likewise for **v3** = 0



If **v2 = v3**

v1 • (v2+v3) = v1 • (v2+v2) = v1 • (2∗v2)

= 2∗(v1•v2)

and (**v1 • v2**) + (**v1 • v3**) = (**v1 • v2**) + (**v1 • v2**)

= 2∗(v1•v2)

If **v2** = – **v3**

v1 • (v2+v3) = v1 • 0

= 0

and (**v1 • v2**) + (**v1 • v3**) = (**v1 • v2**) – (**v1 • v2**)

= 0

If **v1 • v2**) = 0 and **v1 • v3** = 0

v1 • (**v2+v3**) = **v1** • q(**v2+v3**)∗(a∗**u**(**v2**) + b∗**u**(**v3**))

= a∗q(**v2+v3**)∗**v1** • **u**(**v2**)

+ b∗q(**v2+v3**)∗**v1** • **u**(**v3**)

= 0 + 0

and (**v1 • v2**) + (**v1 • v3**) = 0 + 0

Einstein: "Encouraging.

Breton: "But not a proof that in every instance that

v1 • (**v2+v3**) = **v1 • v2 + v1 • v3**.

Would you like to try proving the proposition generally?

Einstein: "Let's try together. We already know they would be equal if

(q(**v2**)∗(**uv1•uv2**) + q(**v3**)∗(**uv1•uv3**)

= **u**(**v1**)•(q(**v2+v3**)∗**u**(**v2+v3**)

Breton: "We already know

q(**v2**)∗(**u**(**v1**)•**u**(**v2**)) = **u**(**v1**)•(q(**v2**)∗**u**(**v2**)

and similarly for **v3.** So we need only be concerned with the sum.

Einstein: "And we know that from our discussion of translated vectors

q(**v2+v3**)∗**u**(**v2+v3**) = q(**v2**)∗**u**(**v2**) + q(**v3**)∗**u**(**v3**).

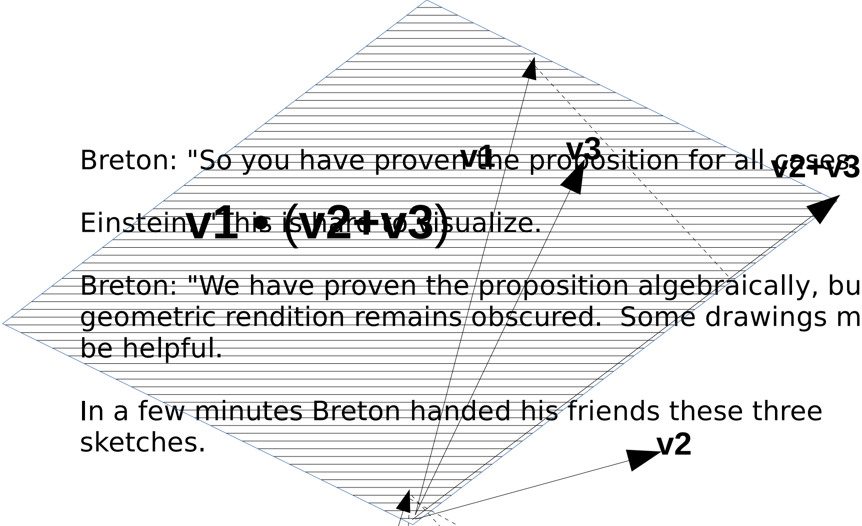
So

u(**v1**)•(q(**v2+v3**)∗**u**(**v2+v3**)

= **u**(**v1**)•(q(**v2**)∗**u**(**v2**) + q(**v3**)∗**u**(**v3**))

= **u**(**v1**)•(q(**v2**)∗**u**(**v2**) + **u**(**v1**)•(q(**v3**)∗**u**(**v3**)

= q(**v2**)∗(**uv1•uv2**) + q(**v3**)∗(**uv1•uv3**)

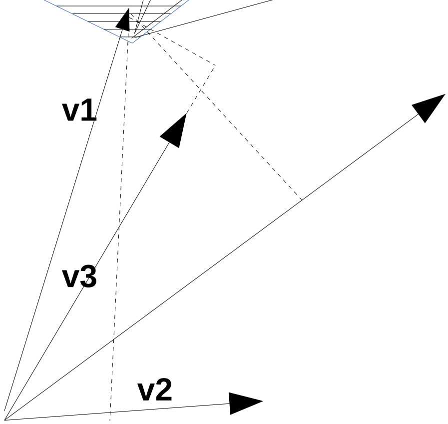


Breton: "So you have proven the proposition for all cases."

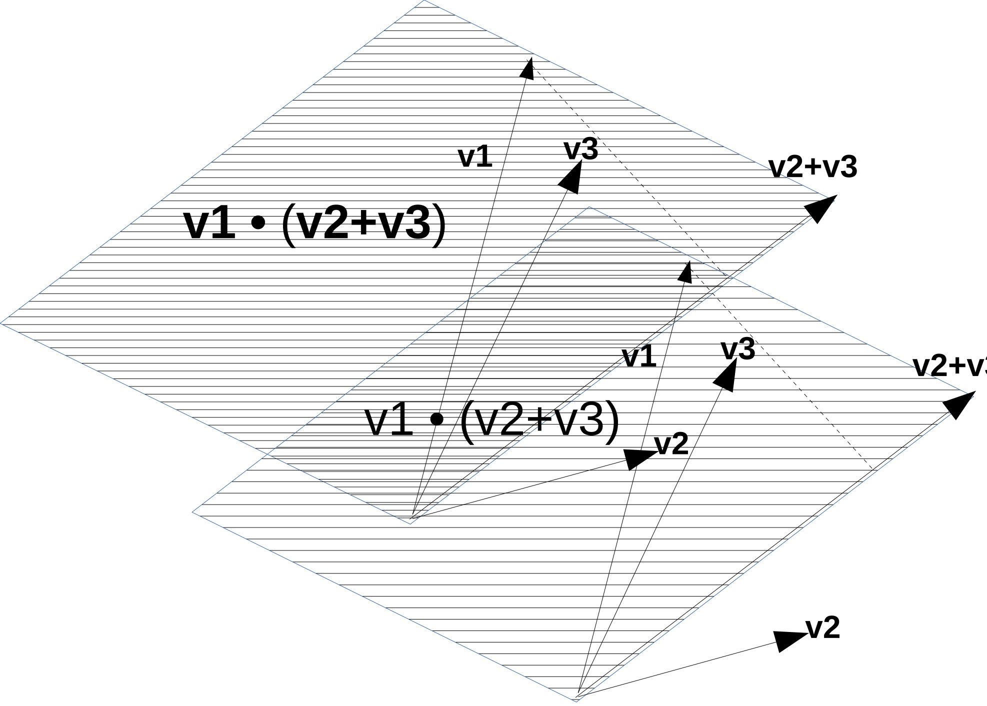
Einstein: "This is hard to visualize."

Breton: "We have proven the proposition algebraically, but the geometric rendition remains obscured. Some drawings may be helpful.

In a few minutes Breton handed his friends these three sketches.

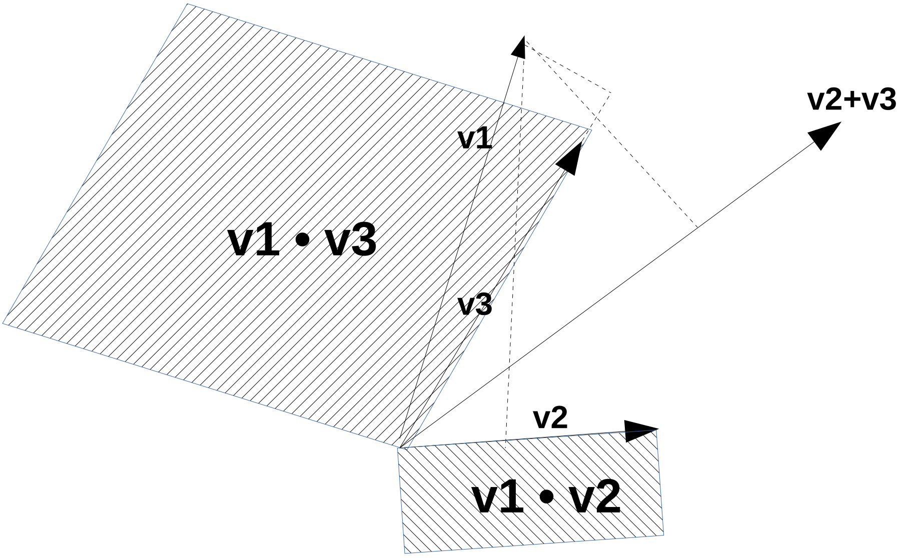


Breton: "This first sketch shows the two vectors, **v2** and **v3** and their sum lying in the same plane. The vector **v1** sticks up from the plane. The dotted lines show the orthogonals from from **v1** to **v2**, **v3**, and **v2+v3**. The orthogonals are related to inner products.

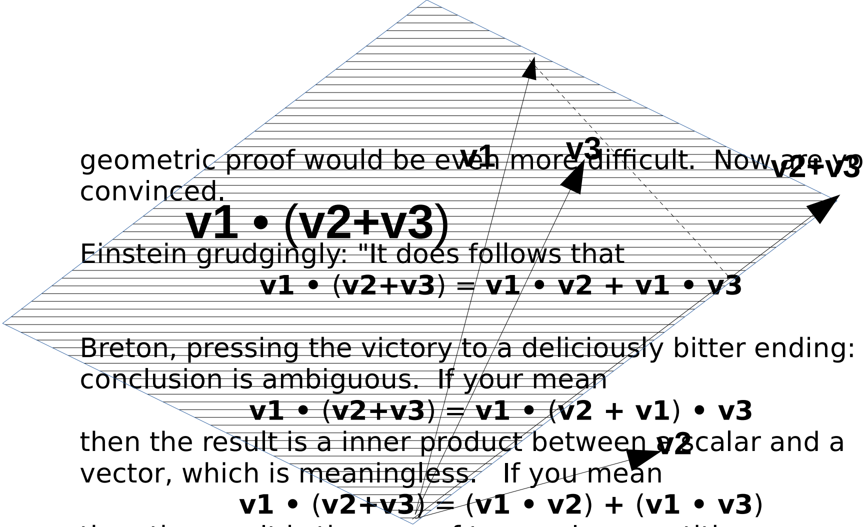


Breton: "Geometrically, this area lies in a plane orthogonal to the plane defined by **v1** and (**v2+v3**).

The next sketch shows the two areas **v1 • v2** and **v1 • v3**.



Breton: "While the algebraic proof requires fine reasoning, the



geometric proof would be even more difficult. Now are you convinced.

Einstein grudgingly: "It does follows that

v1 • (**v2+v3**) = **v1** • **v2** + **v1** • **v3**

Breton, pressing the victory to a deliciously bitter ending: "The conclusion is ambiguous. If your mean

v1 • (**v2+v3**) = **v1** • (**v2** + **v1**) • **v3**

then the result is a inner product between a scalar and a vector, which is meaningless. If you mean

v1 • (**v2+v3**) = (**v1** • **v2**) + (**v1** • **v3**)

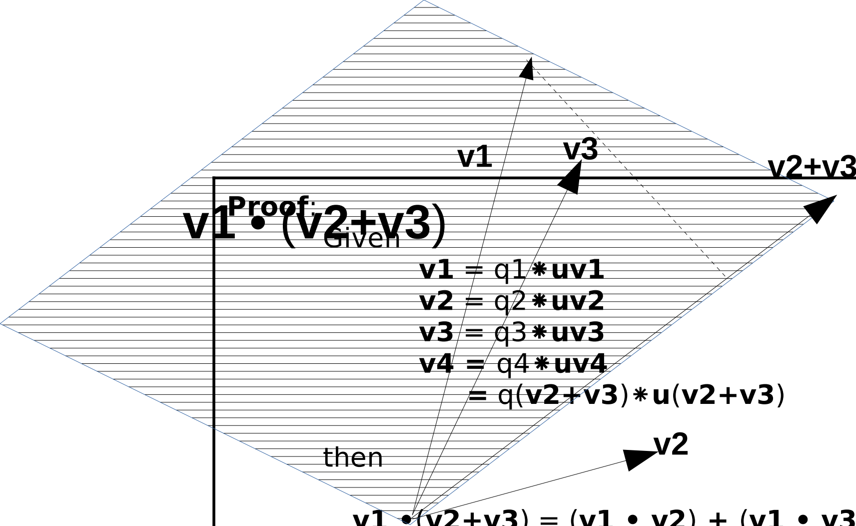
then the result is the sum of two scalar quantities, a meaningful result."

After a short pause Breton continued in an agreeable tone. "Your reasoning follows the format for our formal proofs. Why not use the format we agreed upon? But before that, I suggest we simplify our notation. Let us write

q1 for q(**v1**)
q2 for q(**v2**)
q3 for q(**v3**)
uv1 for **u**(**v1**)
uv2 for **u**(**v2**)
uv3 for **u**(**v3**)

Whenever no ambiguity will follow, we can do the same in other contexts.

Einstein joining gladly: "Agreed. Here's my proof."



Proof:

Given

v1 = q1∗**uv1**
v2 = q2∗**uv2**
v3 = q3∗**uv3**
v4 = q4∗**uv4**
= q(**v2+v3**)∗**u**(**v2+v3**)

then

v1 •(**v2+v3**) = (**v1 • v2**) + (**v1 • v3**)

Proof:
v1 • (**v2+v3**)
= qv1∗**uv1•**(q4∗**uv4**)
= qv1∗**uv1•**(q(**v2+v3**)∗**u**(**v2+v3**)
= qv1∗**uv1•**(qv2∗**uv2** + qv3∗**uv3**)
= qv1∗**uv1•**qv2∗**uv2** + qv1∗**uv1•**qv3∗**uv3**)
=(**v1 • v2**) + (**v1 • v3**)

qed

Breton: "The proof rests on the rhomboid definition of plus in the vector set.

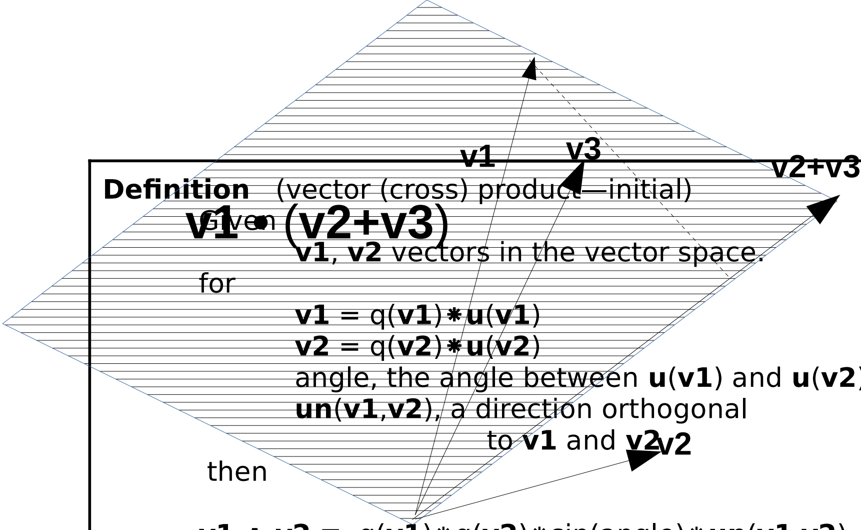
Newton, with a note of delightful satisfaction: "We are building a mathematical structure−−the parts fit together.

Breton: "We've become intellectual carpenters.

Vector product

Newton: " How about multiplication2?

Breton: "Again we need a definition.



Definition (vector (cross) product—initial)

Given

v1, **v2** vectors in the vector space.

for

v1 = q(**v1**)∗**u**(**v1**)
v2 = q(**v2**)∗**u**(**v2**)
angle, the angle between **u**(**v1**) and **u**(**v2**)
un(**v1**,**v2**), a direction orthogonal
to **v1** and **v2**

then

v1 ∧ v2 ≡ q(**v1**)∗q(**v2**)∗sin(angle)∗**un**(**v1**,**v2**)

end of definition

Multiplication2 is called the **cross product**. To avoid confusion with the other multiplications in the vector set it is symbolized with '**∧**'.

As you can see

∧: VxV ⟶ V

Newton: "The vector product depends not only on the angle between the two vectors, but also on an orthogonal direction. We must have

un • v1 = 0

and

un • v2 = 0

Does such a vector exist?"

Breton: "We need only

un • u(**v1**) = 0

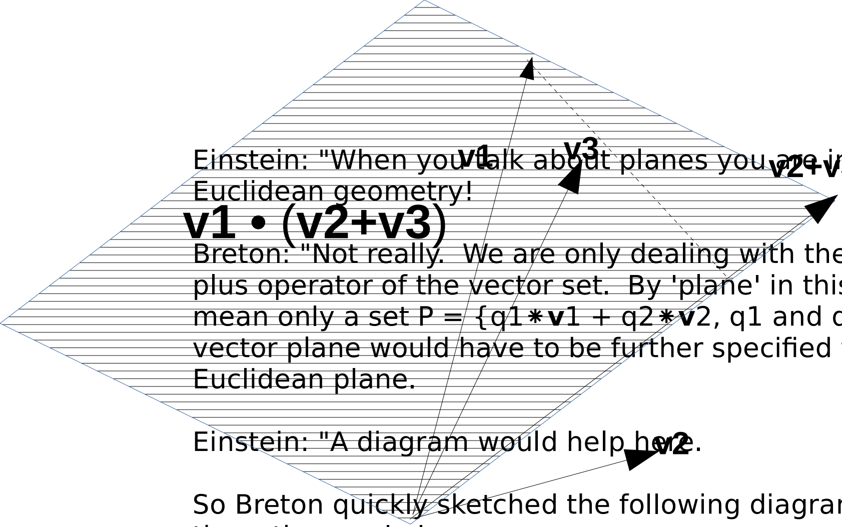
and

un • u(**v2**) = 0

so we need deal only with the unit sphere."

Newton: " Since all directions are part of our vector set, we can find one which is orthogonal to **u**(**v1**).

Breton: "The two vectors, **v1** and **v2**, can be used to define a plane. Since each of the vectors of this plane are orthogonal to **un**, we call this plane **orthogonal** to **un**.

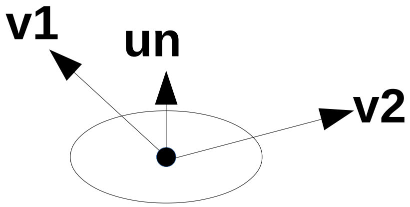


Einstein: "When you talk about planes you are implying Euclidean geometry!

Breton: "Not really. We are only dealing with the axiomatic plus operator of the vector set. By 'plane' in this context I mean only a set P = {q1∗**v**1 + q2∗**v**2, q1 and q2 in Q}. The vector plane would have to be further specified to make it a Euclidean plane.

Einstein: "A diagram would help here.

So Breton quickly sketched the following diagram to illustrate the orthogonal planes.



Breton: "Imagine the disk a unit circle viewed from the side. The circle is a great circle from a unit sphere which you have to imagine. The two vectors **v1** and **v2** lie in a plane which also contains the unit circle. The unit vector **un** is orthogonal to the unit circle and so orthogonal to both **v1** and **v2**.

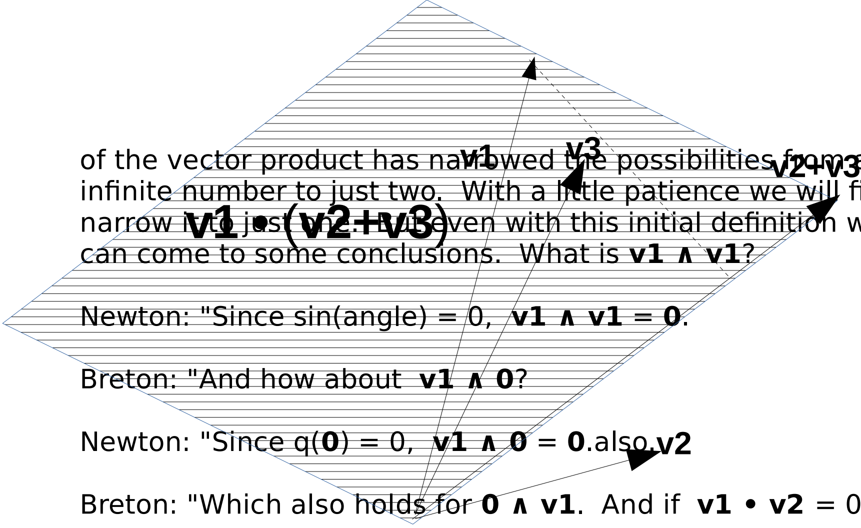
Newton: "So there *does* exist a direction which is orthogonal to both **v1** and **v2**. In fact all directions which lie in the plane of the unit circle are orthogonal to **un**.

Einstein: "If. **un** is orthogonal to **v1** then so also is –**un**. There are thus *two* orthogonal vectors, in opposite directions. So Breton your definition is flawed. You have narrowed the possibilities, but for an adequate definition you would have to narrow the possibilities to only one.

Breton: "True enough. Notice that the definition is only an initial definition. A final definition will be forthcoming.

Einstein: "Promises, promises, always promises.

Breton: "Which will be kept in due time. The initial definition



of the vector product has narrowed the possibilities from an infinite number to just two. With a little patience we will finally narrow it to just one. But even with this initial definition we can come to some conclusions. What is **v1 ∧ v1**?

Newton: "Since sin(angle) = 0, **v1 ∧ v1** = **0**.

Breton: "And how about **v1 ∧ 0**?

Newton: "Since q(**0**) = 0, **v1 ∧ 0** = **0**.also.

Breton: "Which also holds for **0 ∧ v1**. And if **v1 • v2** = 0?

Newton: "Since sin(angle) = 1,

v1 ∧ v2 = q(**v1)**∗q(**v2**)∗**un**(**v1**,**v2**).

Einstein: "Again an ambiguous result.

Breton: "Which will be resolved anon. Notice when the value of the inner product is a minimum, the vector product has its maximum length. Conversely when **v1 ∧ v1** = **0**,

v1 • v1 = q(**v1**)∗q(**v1**)

attains its maximum value.

Einstein: "But when **v1 ∧ 0** = **0, v1 • 0** = 0.

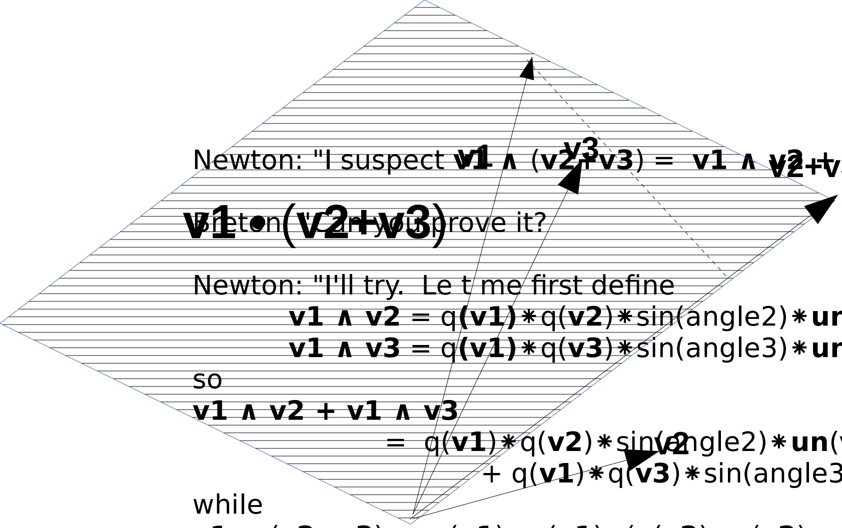
Newton: “We found the inner product interesting, but what possible interest can we expect from the cross product?

Breton: "Look at my diagram again. If **v1** and **v2** are swirling, then an effect could be produced in the orthogonal direction. So to investigate the motion of propellers, we might find the cross product useful.

Einstein: "And for electricity as well.

Breton: "Interesting prospects, don't you think Newton? But let us focus again on our mathematical aim of defining an algebra for the set of vectors. We have not finished with vector multiplication. What can we say about **v1 ∧** (**v2+v3**)?

Sums of vector products



Newton: "I suspect **v1 ∧** (**v2+v3**) = **v1 ∧ v2 + v1 ∧ v3**.

Breton: "Can you prove it?

Newton: "I'll try. Le t me first define

v1 ∧ v2 = q**(v1)**∗q(**v2**)∗sin(angle2)∗**un**(**v1**,**v2**)
v1 ∧ v3 = q**(v1)**∗q(**v3**)∗sin(angle3)∗**un**(**v1**,**v3**)

so

v1 ∧ v2 + v1 ∧ v3
= q(**v1**)∗q(**v2**)∗sin(angle2)∗**un**(**v1**,**v2**)
+ q(**v1**)∗q(**v3**)∗sin(angle3)∗**un**(**v1**,**v3**))

while

v1 ∧ (**v2+v3**) = q(**v1**)∗**u**(**v1**)**∧**(q(**v2**)∗**u**(**v2**) + q(**v3**)∗**u**(**v3**))
= **u**(**v1**)**∧**(q(**v1**)∗q(**v2**)∗**u**(**v2**)
+ q(**v1**)∗q(**v3**)∗**u**(**v3**))
= q(**v1**)∗q(**v2**)∗**u**(**v1**)**∧u**(**v2**)
+ q(**v1**)∗q(**v3**)∗**u**(**v1**)**∧u**(**v3**))
= q(**v1**)∗q(**v2**)∗sin(angle2)∗**un**(**v1**,**v2**)
+ q(**v1**)∗q(**v3**)∗sin(angle3)∗**un**(**v1**,**v3**))

So they *are* equal.

Breton: "Would you put your reasoning into a formal proof?

Newton: "Try this."

Proof:

Given

v1 = q1∗**uv1**
v2 = q2∗**uv2**
v3 = q3∗**uv3**
v4 = q4∗**uv4**
= q(**v2+v3**)∗**u**(**v3+v4**)

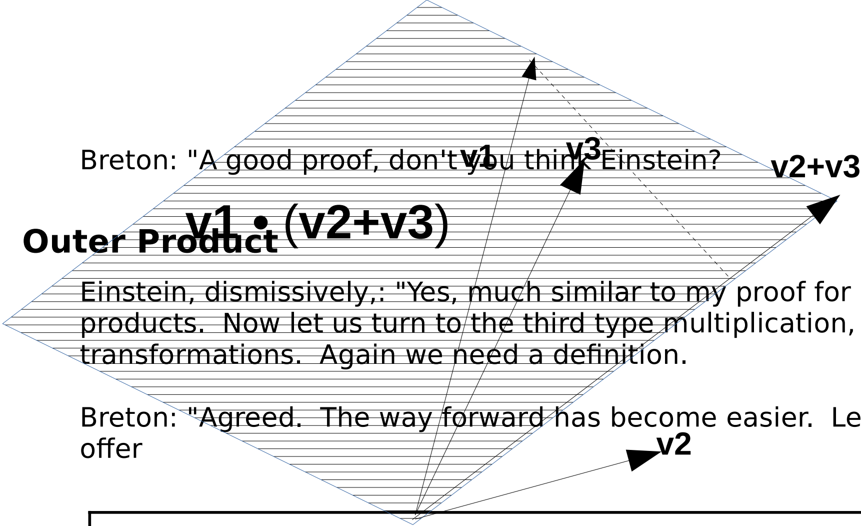
then

v1 ∧ (**v2+v3**) = (**v1 ∧ v2**) **+** (**v1 ∧ v3**)

Proof:

v1 ∧ (**v2+v3**)
= qv1∗**uv1∧**(q4∗**uv4**)
= qv1∗**uv1∧**(q(**v2+v3**)∗**u**(**v3+v4**))
= qv1∗**uv1∧**(qv2∗**uv2 +** qv3∗**uv3**)
= qv1∗**uv1∧**qv2∗**uv2 +** qv1∗**uv1∧**qv3∗**uv3**)
=(**v1 ∧ v2**) **+** (**v1 ∧ v3**)

qed



Breton: "A good proof, don't you think Einstein?

Outer Product

Einstein, dismissively,: "Yes, much similar to my proof for inner products. Now let us turn to the third type multiplication, transformations. Again we need a definition.

Breton: "Agreed. The way forward has become easier. Let me offer

> **Definition** (outer product)
> Given
> **v1**, **v2, v3** vectors in the vector space.
>
> **v3**•[**v1 ⁕ v2**] ≡ (**v3**•**v1**) ⁕ **v2**
>
> end of definition

As you can see the transformation [**v1 ⁕ v2**] transforms the vector **v3** into a scaled vector parallel to **v2**. Now tell me: ‘Is the transformation [**v1 ⁕ v2**] identical with [**v2 ⁕ v1**]?

Einstein: "Of course not! The transformation [**v1 ⁕ v2**] operates to produce a vector parallel to **v2** while [**v2 ⁕ v1**] operates to produce a vector parallel to **v1**

Newton: "Then although

v1 • v2 = **v2 • v1**

still

[**v1 ⁕ v2**] ≠ [**v2 ⁕ v1**]

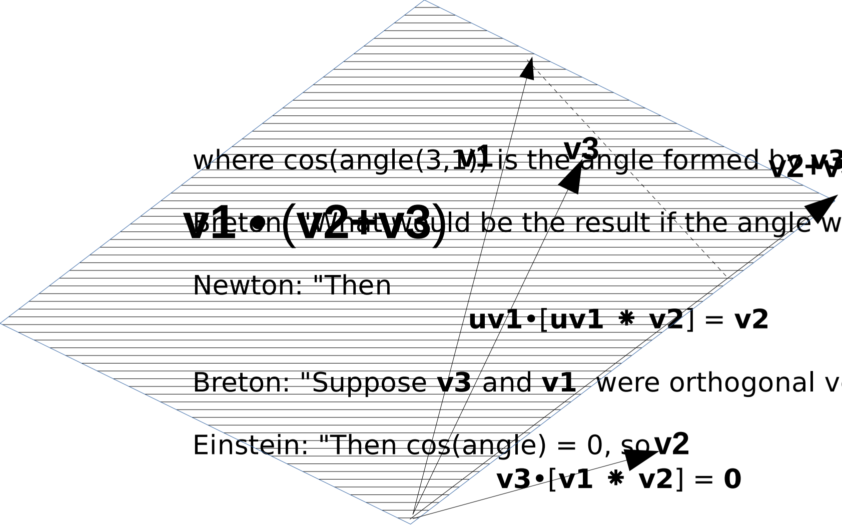
Breton: "Which confirms the difference between these multiplications. In summary

v1 • v2 = **v2 • v1**
v1 ∧ v2 = −**v2 ∧ v1**
[**v1 ⁕ v2**] ≠ [**v2 ⁕ v1**]

Suppose **v3** and **v1** are unit vectors. What would be the result?

Newton: "Then

uv3•[**uv1 ⁕ v2**] = cos(angle(3,1))⁕ **v2**



where cos(angle(3,1)) is the angle formed by **v3** and **v1**.

Breton: "What would be the result if the angle were 0?

Newton: "Then

uv1•[**uv1 ⁕ v2**] = **v2**

Breton: "Suppose **v3** and **v1** were orthogonal vectors.

Einstein: "Then cos(angle) = 0, so

v3•[**v1 ⁕ v2**] = **0**

Breton: "And if the angle equals 180 degrees?

Einstein: "Then cos(angle) = –1, so

v3•[**v1 ⁕ v2**] = –**v2**

Breton: "For given vectors, even though the multiplications are distinct, a certain symmetry appears in their ranges. Here let me illustrate by this table.

PRODUCT	RANGE
v1 • v2	q(**v1**)∗q(**v2**) to –q(**v1**)∗q(**v2**)
v1 ∧ v2	q(**v1**)∗q(**v2**)**⁕un** to –q(**v1**)∗q(**v2**)**⁕un**
v3•[**v1 ⁕ v2**]	q(**v1**)∗q(**v3**)**⁕v2** to –q(**v1**)∗q(**v3**)**⁕v2**

Sums of outer products

Breton: "What can we say about

v3•[**v1 ⁕ v2**] **+ v3**•[**v4 ⁕ v5**]?

Newton: "That's easy.

v3•[**v1 ⁕ v2**] = (**v3**•**v1**) **⁕ v2**

and

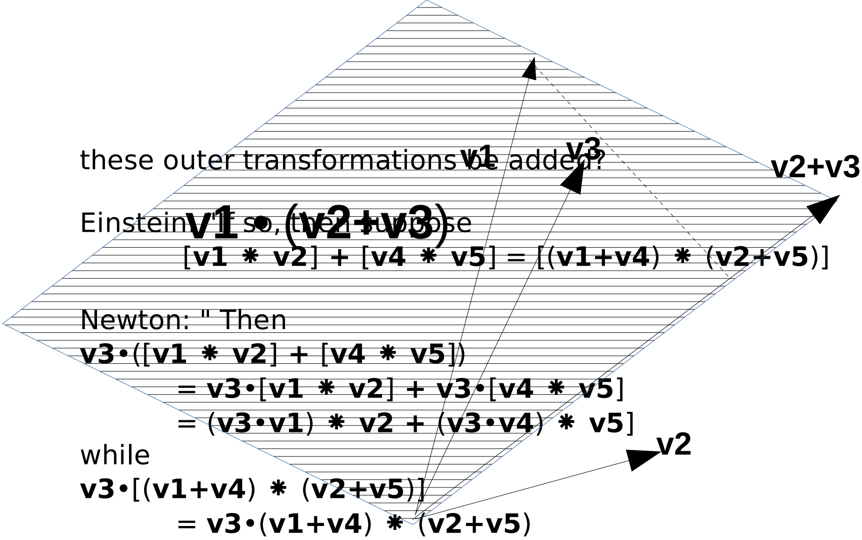
v3•[**v4 ⁕ v5**] = (**v3**•**v4**) **⁕ v5**

so

v3•[**v1 ⁕ v2**] **+ v3**•[**v4 ⁕ v5**]

= (**v3**•**v1**) **⁕ v2 +** (**v3**•**v4**) **⁕ v5**

Breton: "So the outer multiplication is not so mysterious! Can



these outer transformations be added?

Einstein: "If so, then suppose

[**v1** ⁕ **v2**] + [**v4** ⁕ **v5**] = [(**v1**+**v4**) ⁕ (**v2**+**v5**)]

Newton: " Then

v3•([**v1** ⁕ **v2**] + [**v4** ⁕ **v5**])

= **v3**•[**v1** ⁕ **v2**] + **v3**•[**v4** ⁕ **v5**]

= (**v3**•**v1**) ⁕ **v2** + (**v3**•**v4**) ⁕ **v5**]

while

v3•[(**v1**+**v4**) ⁕ (**v2**+**v5**)]

= **v3**•(**v1**+**v4**) ⁕ (**v2**+**v5**)

= (**v3**•**v1**+**v3**•**v4**) ⁕ (**v2**+**v5**)

= (**v3**•**v1**+**v3**•**v4**) ⁕ **v2**

+ (**v3**•**v1**+**v3**•**v4**) ⁕ **v5**

so it does *not* appear that outer products can be added.

Breton: "Not as outer products, but perhaps the sum of two outer products can result in another kind of transformation.

Einstein: "Possibly. Breton, you raise an interesting possibility. Since the outer product transforms one vector into another, perhaps the cross product which also yields a vector different from each multiplicand can also be expressed as a transformation.

Newton, concerned about becoming defocused: "Before we wander off, let's stick to the trail of outer products.

Combinations of Multiplications

Breton: "Right. Let's investigate combinations of these multiplications.

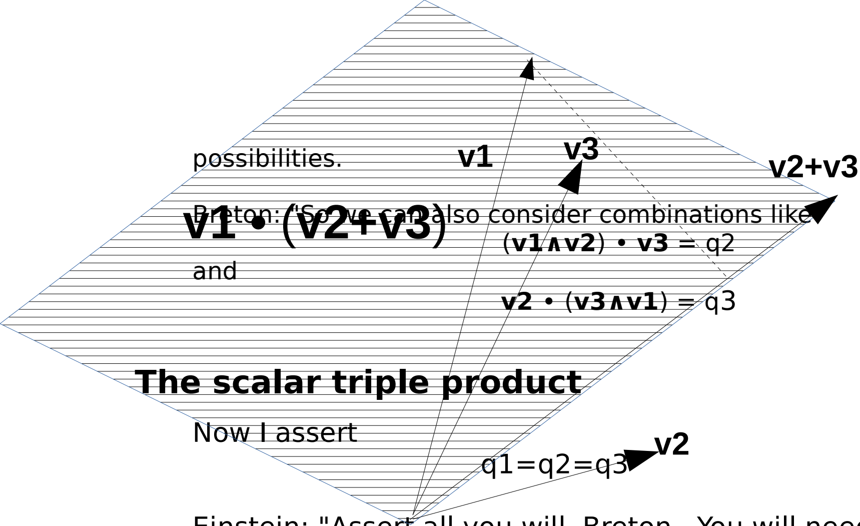
Newton: "What? These multiplications can be combined?

Breton: "Why not? The cross product of two vectors yields a vector which can then be a multiplicand of the inner product with a third vector to produce a quotient number. So isn't a combination like

v1 • (**v2**∧**v3**) = q1

legitimate?

Einstein: "Of course. Such combinations open interesting



possibilities.

Breton: "So we can also consider combinations like

$$(\mathbf{v1} \wedge \mathbf{v2}) \bullet \mathbf{v3} = q2$$

and

$$\mathbf{v2} \bullet (\mathbf{v3} \wedge \mathbf{v1}) = q3$$

The scalar triple product

Now I assert

$$q1 = q2 = q3$$

Einstein: "Assert all you will, Breton. You will need to prove it before I accept it.

Breton: "It does seem astounding, you are right to question. If what I assert is true, we will have mounted a little higher up the mountain of our adventure from which we might expect to open up to a large panoramic vista.

Let me start by making the proof a little easier. Defining

$$\mathbf{v1} \equiv qv1 * \mathbf{uv1}$$
$$\mathbf{v2} \equiv qv2 * \mathbf{uv2}$$
$$\mathbf{v3} \equiv qv3 * \mathbf{uv3}$$

then

$\mathbf{v1} \bullet (\mathbf{v2} \wedge \mathbf{v3})$

$$= \mathbf{v1} \bullet (qv2 * qv3 * \sin(angle23) * \mathbf{un23})$$
$$= qv1 * \mathbf{uv1} \bullet (qv2 * qv3 * \sin(angle23) * \mathbf{un23})$$
$$= qv1 * qv2 * qv3 * \sin(angle23) * \mathbf{uv1} \bullet \mathbf{un23}$$
$$= qv1 * qv2 * qv3 * \sin(angle23) * \cos(angle(\mathbf{v1,un23}))$$

Likewise,

$\mathbf{v2} \bullet (\mathbf{v3} \wedge \mathbf{v1})$

$$= qv1 * qv2 * qv3 * \sin(angle31) * \cos(angle(\mathbf{v2,un31}))$$

$\mathbf{v3} \bullet (\mathbf{v1} \wedge \mathbf{v2})$

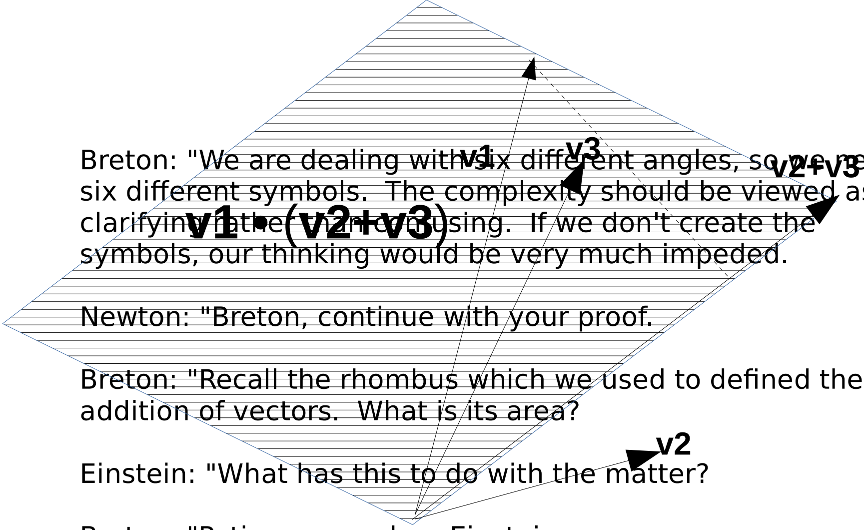
$$= qv1 * qv2 * qv3 * \sin(angle12) * \cos(angle(\mathbf{v3,un12}))$$

The factor $qv1 * qv2 * qv3$ appears in all three equations where they form equal products.

So we need only consider whether

$$\sin(angle23) * \cos(angle(\mathbf{v1,un23}))$$
$$= \sin(angle31) * \cos(angle(\mathbf{v2,un31}))$$
$$= \sin(angle12) * \cos(angle(\mathbf{v3,un12}))$$

Einstein: "All these new definitions can be confusing.



Breton: "We are dealing with six different angles, so we need six different symbols. The complexity should be viewed as clarifying rather than confusing. If we don't create the symbols, our thinking would be very much impeded.

Newton: "Breton, continue with your proof.

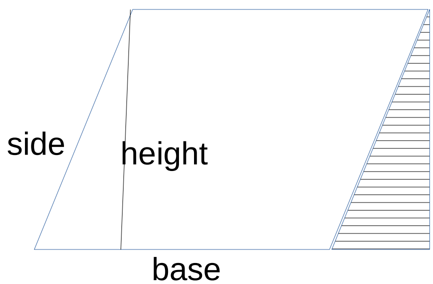
Breton: "Recall the rhombus which we used to defined the addition of vectors. What is its area?

Einstein: "What has this to do with the matter?

Breton: "Patience, my dear Einstein.

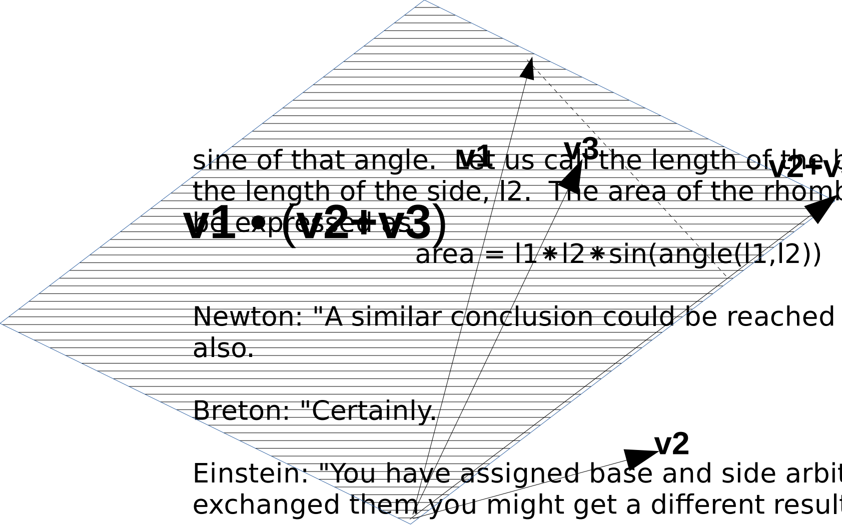
Newton: "Everyone know the area of a rhombus is the product of its base with its height.

Breton: "Not necessarily then, the product of its base with its *side*. Let me illustrate. With that Breton drew the following illustration.



Area of a rhombus

Breton: "The area of the rhombus equals the product of its base and height because one can translate the triangle with the slanted side to the other side of the rhombus, the area I have indicated by the hatched triangle, to produce a square. Subtracting the area of one triangle and replacing with the area of ab equal triangle does not change the area. The reconstructed area is then a square whose area is clearly the product of the length of its base with the length of its height.
 Now consider the angle between the base and the side. The length of the height is then the length of the side times the



sine of that angle. Let us call the length of the base, l1, and the length of the side, l2. The area of the rhombus can then be expressed as:

$$\text{area} = l1 * l2 * \sin(\text{angle}(l1,l2))$$

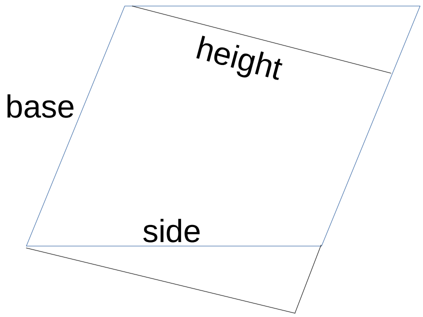
Newton: "A similar conclusion could be reached for rhomboids also.

Breton: "Certainly.

Einstein: "You have assigned base and side arbitrarily. If you exchanged them you might get a different result.

Newton: "I'm beginning to see how this argument could lead to proving your contention, Breton. But show us how it makes no difference which side is considered the base.

Breton: "It's not clear from the illustration? All right, let's switch the sides. Here is a second illustration,



So you see the first side has become a new base with a different height, but the same area. So we can conclude

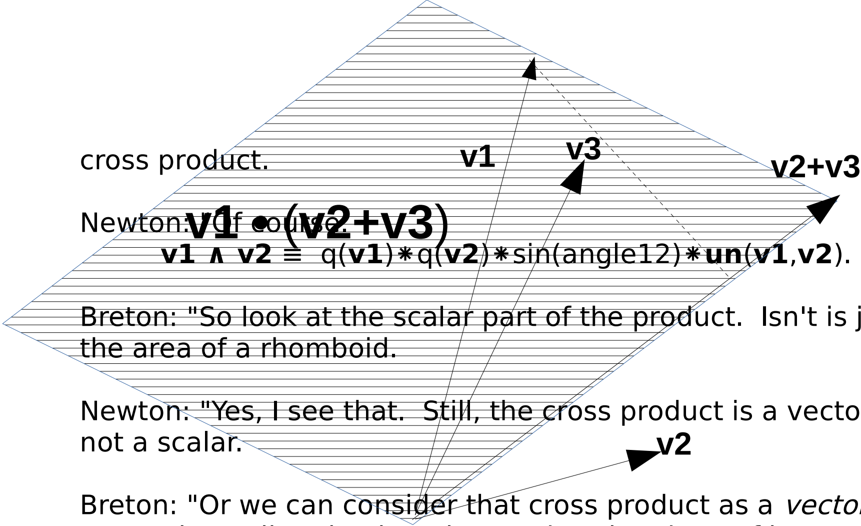
$$\begin{aligned}\text{area} &= l1 * l2 * \sin(\text{angle}(l1,l2)) \\ &= l2 * l1 * \sin(\text{angle}(l2,l1))\end{aligned}$$

So we can further conclude

$$\sin(\text{angle}(l1,l2)) = \sin(\text{angle}(l2,l1))$$

Newton: "What does that mean for vectors?

Breton: "Let the sides and their directions be represented as vectors, **v1** and **v2**. Do you remember the definition of the



cross product.

Newton: "Of course.

$\mathbf{v1} \wedge \mathbf{v2} \equiv q(\mathbf{v1})*q(\mathbf{v2})*\sin(\text{angle}12)*\mathbf{un}(\mathbf{v1},\mathbf{v2})$.

Breton: "So look at the scalar part of the product. Isn't is just the area of a rhomboid.

Newton: "Yes, I see that. Still, the cross product is a vector and not a scalar.

Breton: "Or we can consider that cross product as a *vector* area, whose direction is orthogonal to the plane of its two vectors, and whose length equals the area of the rhomboid defined by the two vectors.

Einstein, mockingly: "So length equals area?

Breton: "I stand corrected. I should have said 'whose *magnitude* equals the area of the rhomboid'.

Einstein: "Much better. So just as the product of two lengths is a scalar area, the cross product of two vectors is a vector area.

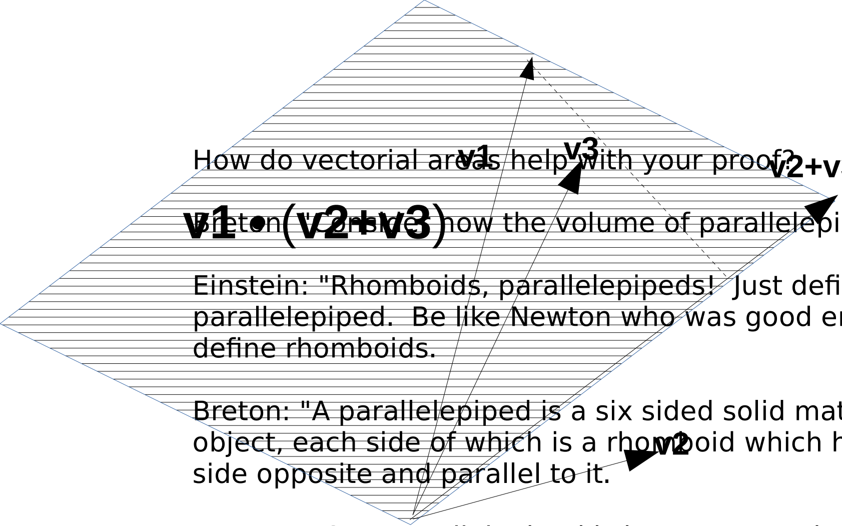
Newton: "Wonderful. The language of vectors subsumes ordinary arithmetic and even surpasses it.

Breton: "In this instance, we now can consider areas as scalars or as vectors. Vectorial language is like singing a song rather than just reading the score.

Einstein: "Remarkable and surprising, but enough of metaphors! We know how to calculate the arithmetical area of a rhomboid and that it makes no difference which side is taken as base, but for vectorial areas, how do we know that either option has the same direction?

Breton: "That's easy enough. Since the two vectors define a plane, any vector orthogonal to both vectors will be orthogonal to the plane, and thus be parallel vectors. The order of the vectors, however, becomes important, since a reversed order will produce a negatively parallel vector.

Einstein: "Breton, you still have not proven your assertion.



How do vectorial areas help with your proof?

Breton: "Consider how the volume of parallelepipeds.

Einstein: "Rhomboids, parallelepipeds! Just define a parallelepiped. Be like Newton who was good enough to define rhomboids.

Breton: "A parallelepiped is a six sided solid mathematical object, each side of which is a rhomboid which has an similar side opposite and parallel to it.

Newton: "So a parallelepiped is just an extension of rhomboids to three dimensions. Since each surface of the parallelepiped is a rhomboid we now know how to calculate its surface area.

Einstein: "So a parallelepiped is just a box.

Newton: "Which may be scrunched up a bit.

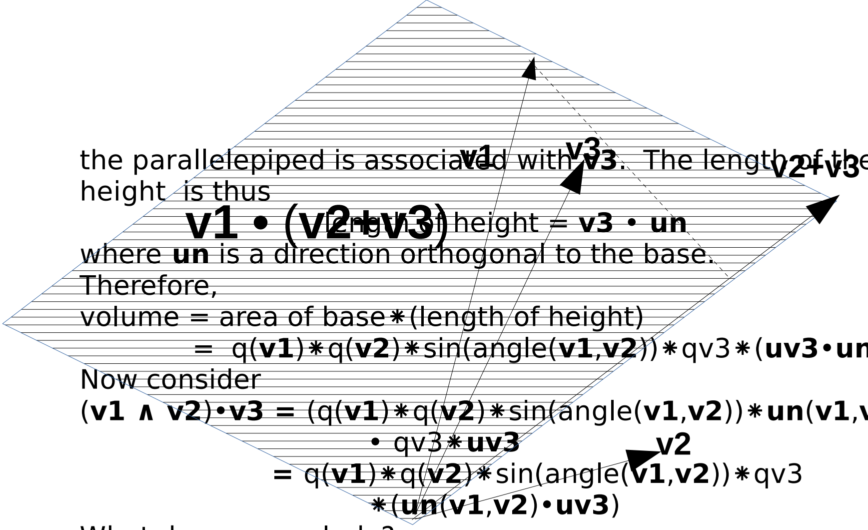
Breton: "We were able to calculate the area of a rhomboid from the knowledge of its sides. Because of parallelism we needed to know only two different sides.
Now parallelepipeds have a volume. How can we calculate its volume?

Newton: "For a rectangular parallelepiped, the answer is the area its base area times its height.

Breton: "Since the area of the base for the rectangular case is just the product of the length of its sides, the volume of the rectangular parallelepiped is just the product of the length of its three edges. More generally, If the base area is a rhomboid, while the remaining edge is perpendicular to the base, the volume of the parallelepiped would be the area of the base rhomboid times the length of the remaining side.

Einstein: "How about the the general case where the remaining side is canted in an arbitrary direction with respect to the base.

Breton: "So let us go vectorial. Let each of the three non–parallel edges be designated **v1**, **v2**, and **v3**. Further let **v1** and **v2** be associated with the base area. Then the height of



the parallelepiped is associated with **v3**. The length of the height is thus

length of height = **v3** • **un**

where **un** is a direction orthogonal to the base.
Therefore,
volume = area of base∗(length of height)
= q(**v1**)∗q(**v2**)∗sin(angle(**v1**,**v2**))∗qv3∗(**uv3**•**un**)

Now consider
(**v1 ∧ v2**)•**v3 =** (q(**v1**)∗q(**v2**)∗sin(angle(**v1**,**v2**))∗**un**(**v1**,**v2**))
• qv3∗**uv3**
= q(**v1**)∗q(**v2**)∗sin(angle(**v1**,**v2**))∗qv3
∗(**un**(**v1**,**v2**)•**uv3**)

What do you conclude?

Newton: "Since **un**(**v1**,**v2**) and **un** are both directions orthogonal to the base, the equations are the same.

Breton: "Not the same, but equal. So (**v1 ∧ v2**)•**v3** equals the volume of a paralellepiped formed by the three vectors.

Einstein: "Nice, but this does not fully prove your assertion. You must further show that any combination of the vectors yields the same result.

Breton: "Fair enough. First do you agree

(**v1 ∧ v2**)•**v3 = v3**•(**v1 ∧ v2**)?

Einstein: "Certainly. As we saw earlier, inner products commute.

Breton: "So now I must show

(**v1 ∧ v2**)•**v3 =**(**v2 ∧ v3**)•**v1**

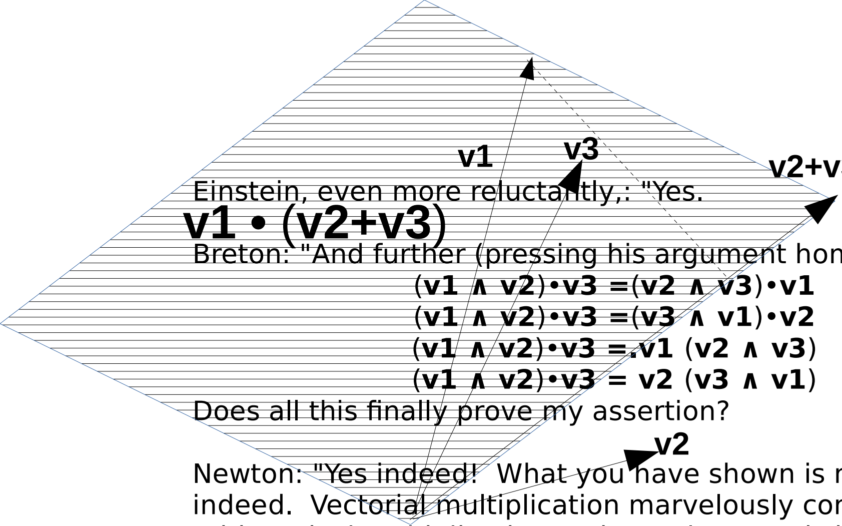
Both Newton and Einstein lean forward eagerly.

Breton: " Now (**v2 ∧ v3**)•**v1** corresponds to a different side of the parallelepiped with a different height. But since it is the same parallelepiped, tell me does it have the same volume?

Einstein: "Yes.

Breton: "So even though the multiplicative factors are different, as volumes

(**v1 ∧ v2**)•**v3 =**(**v2 ∧ v3**)•**v1**



Einstein, even more reluctantly,: "Yes.

Breton: "And further (pressing his argument home)

$$(\mathbf{v1} \wedge \mathbf{v2})\cdot\mathbf{v3} = (\mathbf{v2} \wedge \mathbf{v3})\cdot\mathbf{v1}$$
$$(\mathbf{v1} \wedge \mathbf{v2})\cdot\mathbf{v3} = (\mathbf{v3} \wedge \mathbf{v1})\cdot\mathbf{v2}$$
$$(\mathbf{v1} \wedge \mathbf{v2})\cdot\mathbf{v3} = .\mathbf{v1}\ (\mathbf{v2} \wedge \mathbf{v3})$$
$$(\mathbf{v1} \wedge \mathbf{v2})\cdot\mathbf{v3} = \mathbf{v2}\ (\mathbf{v3} \wedge \mathbf{v1})$$

Does all this finally prove my assertion?

Newton: "Yes indeed! What you have shown is marvelous indeed. Vectorial multiplication marvelously comprehends arithmetical multiplication and greatly extends it. I begin to see how our intellectual vistas are being enlarged.

Einstein: "Not so fast. Since,

$$\mathbf{v}\cdot(-\mathbf{v3}) = -\ (\mathbf{v}\cdot\mathbf{v3})$$
$$. \neq \mathbf{v}\cdot\mathbf{v3}$$
$$(\mathbf{v1} \wedge \mathbf{v2})\cdot(-\mathbf{v3}) \neq (\mathbf{v2} \wedge \mathbf{v3})\cdot\mathbf{v1}$$

What's going on here?

Breton: "Very little gets by you Einstein. What is

$$(\mathbf{v1} \wedge \mathbf{v2})\cdot(-\mathbf{v3})?$$

Einstein: "You tell me.

Breton: "If $(\mathbf{v1} \wedge \mathbf{v2})\cdot(\mathbf{v3})$ is the volume of a parallelepiped, then
$(\mathbf{v1} \wedge \mathbf{v2})\cdot(-\mathbf{v3})$ is the volume of a *different* parallelepiped.

Einstein: "With a negative volume!

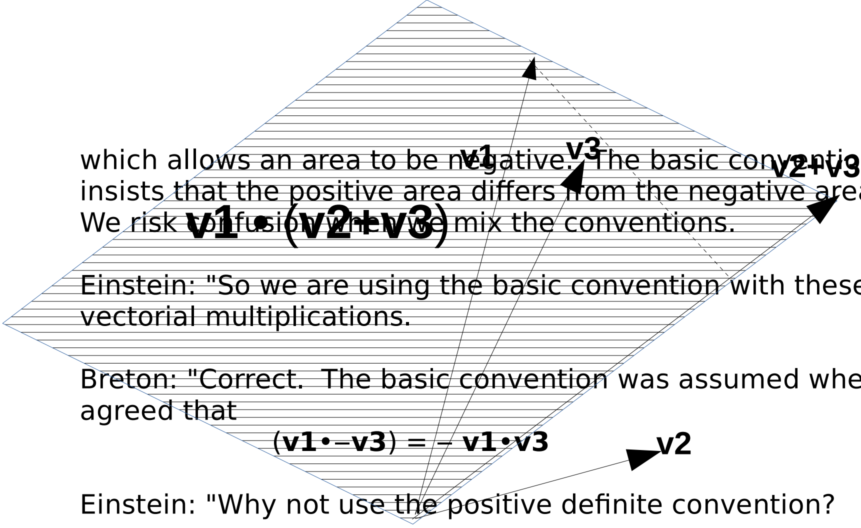
Breton: "Just so. You bring up an important subject which we touched on yesterday. Consider arithmetic multiplication. If the area of a rectangle

$$\text{area} = 2 * 3 = 6$$

what is the area of a rectangle $2 * (-3)$?

Einstein: "How can it be –6?

Breton: "Recall how yesterday we distinguished two conventions: positive definite and basic. If we insist that all areas as positive then we are insisting on the positive definite convention. If not, then we should use the basic convention



which allows an area to be negative. The basic convention insists that the positive area differs from the negative area. We risk confusion when we mix the conventions.

Einstein: "So we are using the basic convention with these vectorial multiplications.

Breton: "Correct. The basic convention was assumed when we agreed that

$$(\mathbf{v1}\bullet-\mathbf{v3}) = -\,\mathbf{v1}\bullet\mathbf{v3}$$

Einstein: "Why not use the positive definite convention?

Newton: "Then we would not be able to use negative numbers.

Breton: "No small restriction. In any case, we have assumed the basic convention.

Einstein: "Then the word 'volume' is misleading.

Breton: "Only from the aspect of the positive definite convention. Our expanded (basic) view allows negative areas as well as negative volumes. This comports well with the possibility that (**v1 ∧ v2**)•(**v3**) may be positive or negative.
  If you insist on the positive definite convention, then we shall have to consider only

$$\mathrm{abs}((\mathbf{v1} \wedge \mathbf{v2})\bullet(\mathbf{v3})).$$

Newton: "The basic convention will do for me.

Einstein: "Then we must see

$$(\mathbf{v1} \wedge \mathbf{v2})\bullet(\mathbf{v3})$$

as different from

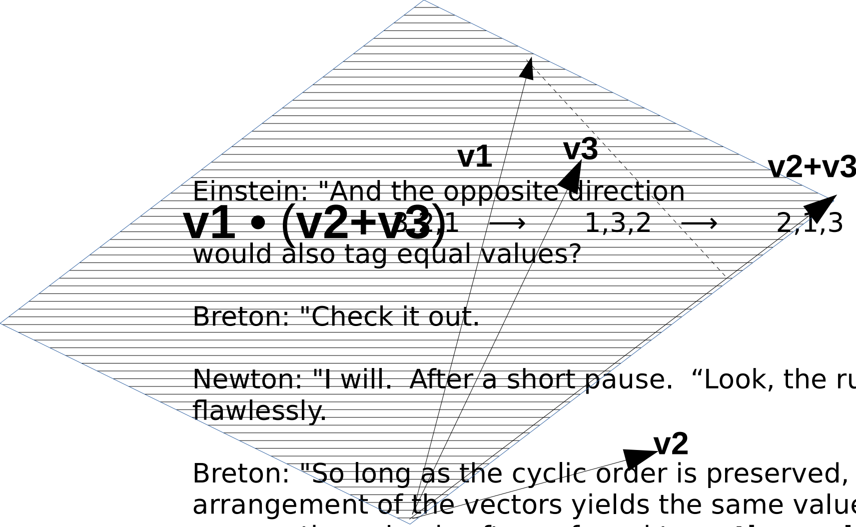
$$(\mathbf{v2} \wedge \mathbf{v1})\bullet(\mathbf{v3})$$

since the second is the negative of the first.

Breton: "Correct. Here's a little memnonic to help associating which volumes are equal. All the equal volumes keep a cyclic order of the vectors. For instance, (**v1 ∧ v2**)•(**v3**) orders the vectors as 1,2,3. It has the same volume as (**v2 ∧ v3**)•(**v1**) which orders the vectors 2,3,1.
  The cyclic order has

$$1,2,3 \longrightarrow 2,3,1 \longrightarrow 3,1,2$$

Vectors so ordered have equal values.



Einstein: "And the opposite direction

3,2,1 $\longrightarrow$ 1,3,2 $\longrightarrow$ 2,1,3

would also tag equal values?

Breton: "Check it out.

Newton: "I will. After a short pause. “Look, the rule works flawlessly.

Breton: "So long as the cyclic order is preserved, any arrangement of the vectors yields the same value. For this reason, the value is often referred to as **the scalar triple product**.

Einstein: "Even though each of the products individually are different, the all have the same value. This is a result I find hard to stomach.

Breton: "We have here another instance of the important distinction between the meaning of the word 'is' and the meaning of the word 'equal'. It is just loose thinking to conflate the two. We could all agree that

$$7+3=4+6=5+5=10$$

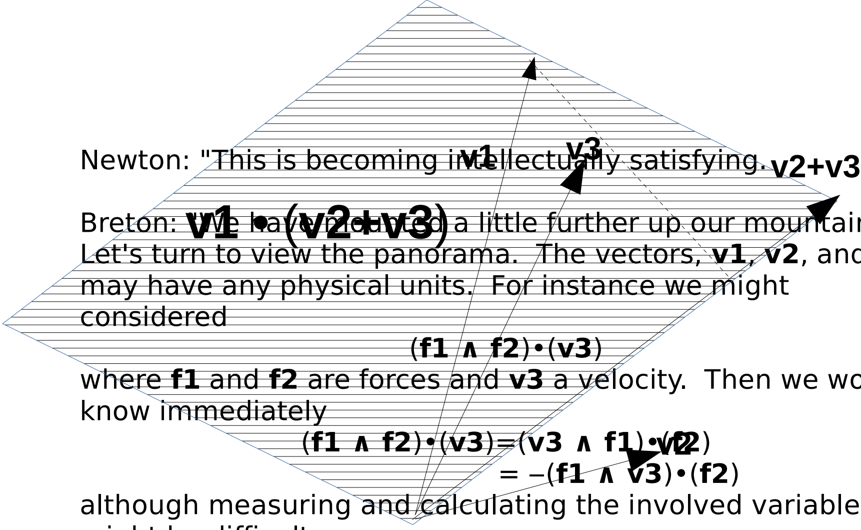
even though {7+3} is not {4+6} which is not {5+5}.

Newton: "Yesterday, you insisted on the same distinction. I agreed then, but now realize better how the difference between 'is' and 'equals' is rooted deeply in our efforts to think correctly.

Breton: "You remind me now of another aspect of triple products which reflects a conclusion reached yesterday. If the vector **v** has units, say L , the the inner, vector, and outer products we have defined should have units, $L*L$, and the triple product units $L*L*L$. In our development taking L as length, then $L*L$ would denote and area and $L*L*L$ a volume.

Newton: "Exactly as we determined.

Breton: "So vectorial algebra fits nicely, at least in this aspect, with our quest for Theoretical Physics.



Newton: "This is becoming intellectually satisfying.

Breton: "We have mounted a little further up our mountain. Let's turn to view the panorama. The vectors, **v1**, **v2**, and **v3** may have any physical units. For instance we might considered

$$(\mathbf{f1} \wedge \mathbf{f2})\cdot(\mathbf{v3})$$

where **f1** and **f2** are forces and **v3** a velocity. Then we would know immediately

$$\begin{aligned}(\mathbf{f1} \wedge \mathbf{f2})\cdot(\mathbf{v3}) &= (\mathbf{v3} \wedge \mathbf{f1})\cdot(\mathbf{f2}) \\ &= -(\mathbf{f1} \wedge \mathbf{v3})\cdot(\mathbf{f2})\end{aligned}$$

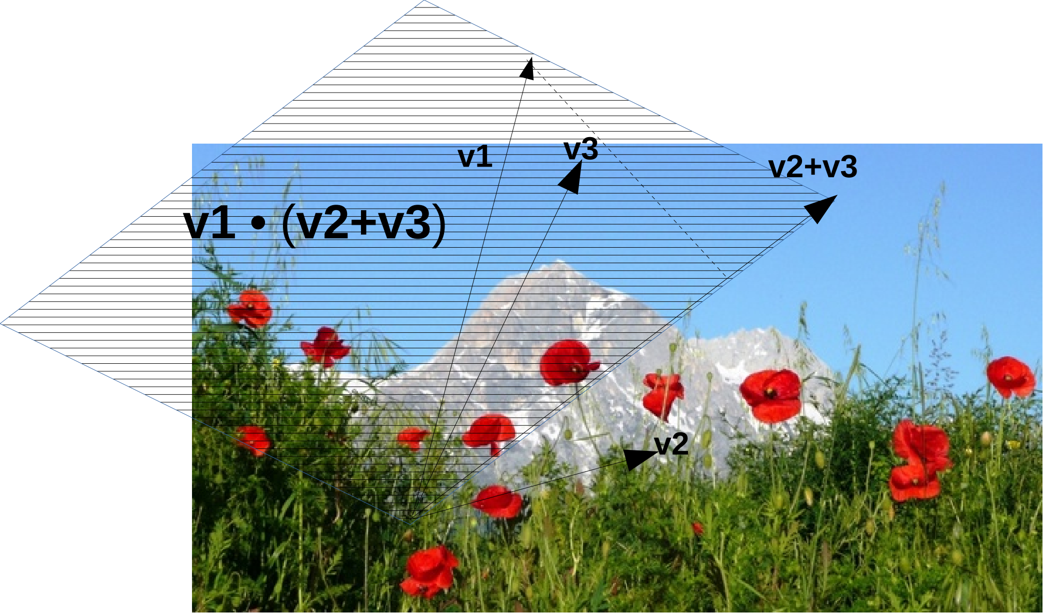
although measuring and calculating the involved variables might be difficult.

Newton, in amazement: "Instead of fashioning only one more intellectual idea for our explorations, we have now a whole warehouse of interesting intellectual tools.

Breton: "I find it pitiful that so much of modern physics is explained merely in terms of scalars. Vectorial explanations offer the prospect of so much more insight. For instance, the idea of area as a scalar has dominated our thinking, but the idea of area can be expanded as a vector, and perhaps even as a transformation. How might these expanded ideas of area enlighten our thinking?

Newton: "Our metaphorical mountain offers more challenges than we first foresaw.

With that, Newton rose from his chair and took down the picture which he had framed during the previous day's conversation.



Breton: "We have extended ourselves. I think it time to regroup and consider our next steps.

Einstein: "I agree. If we climb too fast we can slip and fall. Newton would you kindly summarize for us.

Newton: "Let me start from the axioms themselves. We are given a field of scalar numbers (taken as Q, the quotient numbers) and a set of vectors, and two operators (addition and scalar multiplication) which act on any vector to produce another vector. Let me symbolize then as

$$+:\mathbf{V1xV2} \longrightarrow \mathbf{V3}$$

and

$$*:\mathrm{Q}\mathbf{xV1} \longrightarrow \mathbf{V2}$$

Breton: "You show the sets involved nicely, but please show the symbolism for the action of the operators on individual elements of the vector set, **V**.

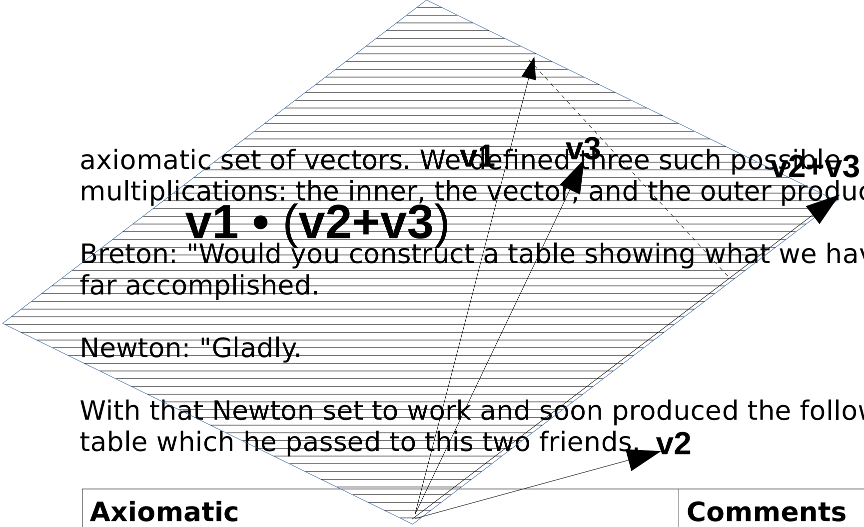
Newton: "Certainly

$$\mathbf{v1+v2} = \mathbf{v3}$$

and

$$q*\mathbf{v1} = \mathbf{v2}$$

The axioms stipulate that these equation always hold.
The vector set, axiomatically, does not define multiplication. So of itself, it is not a field. Breton proposed expanding the vector set to include some other operators, like multiplication and so try to construct a field on the foundation of the



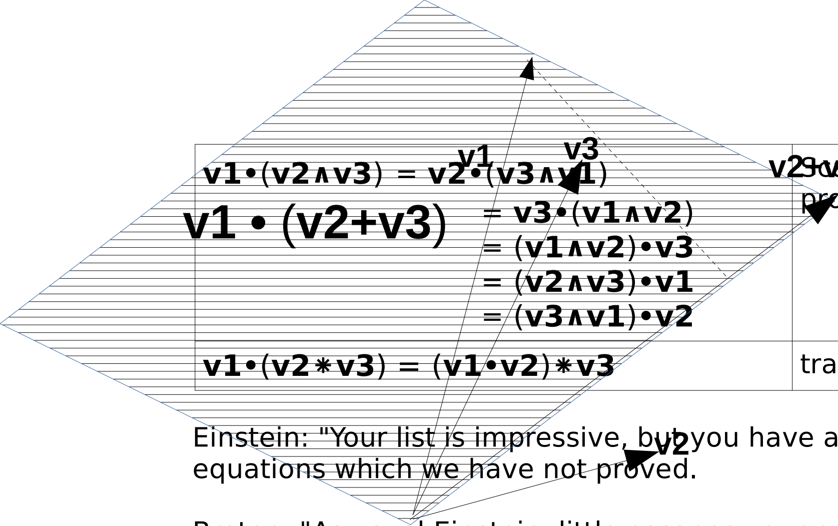
axiomatic set of vectors. We defined three such possible multiplications: the inner, the vector, and the outer products.

Breton: "Would you construct a table showing what we have so far accomplished.

Newton: "Gladly.

With that Newton set to work and soon produced the following table which he passed to this two friends.

Axiomatic	**Comments**
v1+v2 = **v3**	closure
q∗**v1**= **v2**	Scalar multiply
v1+(**v2+v3**) = (**v1+v2**)**+v3**	association
Defined: two at a time	
v1•v2 = **v2•v1**	Inner product
b∗**v1**•c∗**v2** = b∗c∗(**v1•v2**)	
abs(**v1•v2**) ≤ abs(**v1**) ∗ abs(**v2**)	
v1∧v2 = –(**v2∧v1**) = ((–**v2**)∧**v1**) = (**v2∧**(–**v1**))	cross product
v1∧v1 = **0**	
(b∗**v1**)∧(c∗**v2**) = b∗c∗(**v1∧v2**)	
abs(**v1∧v2**) ≤ abs(**v1**)∗abs(**v2**)	
v1•(**v1∧v2**) = **v2•**(**v1∧v2**) = 0	
(b∗**v1**)∗(c∗**v2**) = b∗c∗(**v1∗v2**)	
(abs(**v1**)∗abs(**v2**))² = (abs(**v1∧v2**))² + (abs(**v1•v2**))²	
Defined: three at at time	
v1•(**v2+v3**) = **v1•v2 + v1•v3**	
v1∧(**v2+v3**) = **v1∧v2 + v1∧v3**	
v1∗(**v2+v3**) = **v1∗v2 + v1∗v3**	



$\mathbf{v1}\bullet(\mathbf{v2}\wedge\mathbf{v3}) = \mathbf{v2}\bullet(\mathbf{v3}\wedge\mathbf{v1})$ $= \mathbf{v3}\bullet(\mathbf{v1}\wedge\mathbf{v2})$ $= (\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v3}$ $= (\mathbf{v2}\wedge\mathbf{v3})\bullet\mathbf{v1}$ $= (\mathbf{v3}\wedge\mathbf{v1})\bullet\mathbf{v2}$	Scalar triple product
$\mathbf{v1}\bullet(\mathbf{v2}\ast\mathbf{v3}) = (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3}$	transformation

Einstein: "Your list is impressive, but you have added some equations which we have not proved.

Breton: "As usual Einstein, little escapes your notice. Newton, I see you have added only three such equations. Please tell us why and more importantly prove those assertions.

Newton: "All three equations refer to absolute values. In one

$$\text{abs}(\mathbf{v1}\bullet\mathbf{v2}) \le \text{abs}(\mathbf{v1}) \ast \text{abs}(\mathbf{v2})$$

if we use basic convention

$$\text{abs}(\mathbf{v1}) = \text{abs}(qv1)$$
$$\text{abs}(\mathbf{v2}) = \text{abs}(qv2)$$
$$\text{abs}(\mathbf{v1}\bullet\mathbf{v2}) = \text{abs}(qv1\ast qv2\ast\cos(\text{angle}))$$

The result follows since abs(cos(angle)) ≤ 1.

The second such equation

$$\text{abs}(\mathbf{v1}\wedge\mathbf{v2}) \le \text{abs}(\mathbf{v1}) \ast \text{abs}(\mathbf{v2})$$

follows almost immediately since

$$\text{abs}(\mathbf{v1}\wedge\mathbf{v2}) = \text{abs}(qv1\ast qv2\ast\sin(\text{angle}))$$

since again abs(sin(angle)) ≤ 1.

The third such equation

$$(\text{abs}(\mathbf{v1})\ast\text{abs}(\mathbf{v2}))^2 = (\text{abs}(\mathbf{v1}\wedge\mathbf{v2}))^2 + (\text{abs}(\mathbf{v1}\bullet\mathbf{v2}))^2$$

follows closely.

Note

$$(\text{abs}(\mathbf{v1})\ast\text{abs}(\mathbf{v2}))^2 = (\text{abs}(qv1)\ast\text{abs}(qv2))^2$$
$$\text{abs}(\mathbf{v1}\wedge\mathbf{v2}))^2 = (\text{abs}(qv1)\ast\text{abs}(qv2)\ast\sin(\text{angle}))^2$$
$$\text{abs}(\mathbf{v1}\bullet\mathbf{v2}))^2 = (qv1\ast qv2\ast\cos(\text{angle}))^2$$

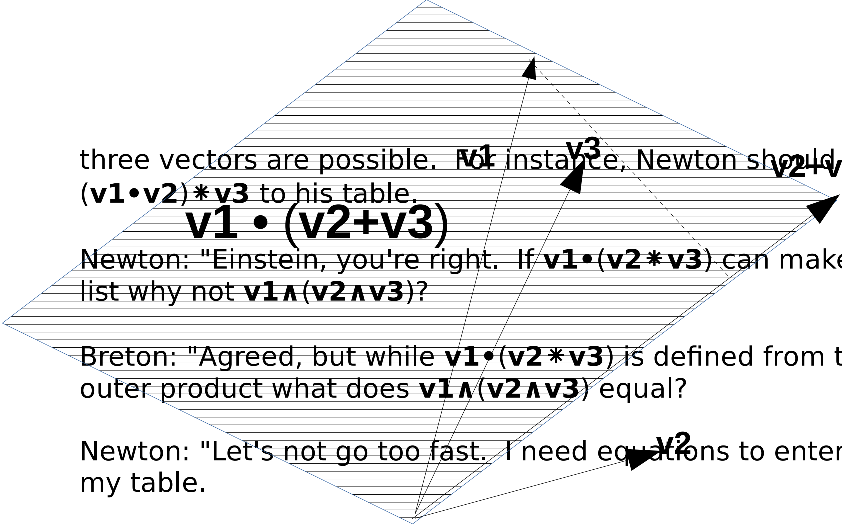
so

$$(\text{abs}(\mathbf{v1}\wedge\mathbf{v2}))^2 + (\text{abs}(\mathbf{v1}\bullet\mathbf{v2}))^2$$
$$= (qv1\ast qv2)^2\ast(\sin^2(\text{angle})+ \ast\cos^2(\text{angle}))$$
$$= (qv1\ast qv2)^2.$$

Breton: "So this third equation simply rests on the identity

$$\sin^2(\text{angle})+ \ast\cos^2(\text{angle}) = 1$$

Einstein: "Good, but it seems to me other combinations of



three vectors are possible. For instance, Newton should add (**v1•v2**)∗**v3** to his table.

Newton: "Einstein, you're right. If **v1•**(**v2∗v3**) can make the list why not **v1∧**(**v2∧v3**)?

Breton: "Agreed, but while **v1•**(**v2∗v3**) is defined from the outer product what does **v1∧**(**v2∧v3**) equal?

Newton: "Let's not go too fast. I need equations to enter into my table.

Einstein: "From the definition of outer product we know

v1•(**v2∗v3**) = (**v1•v2**)∗**v3**

Breton: "We also know from in inner product that

v1•v2 = **v2•v1**

so that

v1•(**v2∗v3**) = (**v2•v1**)∗**v3**.

Einstein: "And again from the outer product

(**v2•v1**)∗**v3** = (**v2•**(**v1∗v3**)

Newton: "Good. I'll add to my table

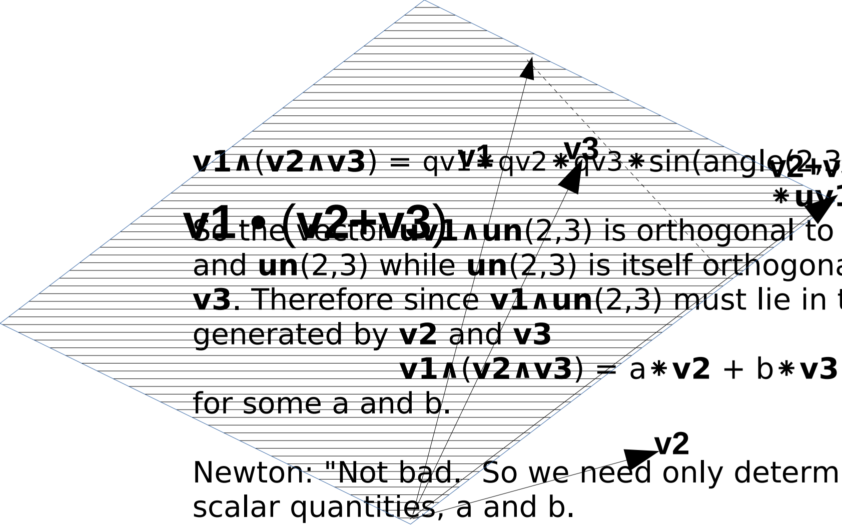
v1•(**v2∗v3**) = (**v1•v2**)∗**v3**
= (**v2•v1**)∗**v3**
= **v2**•(**v1∗v3**)

The vector triple product

Breton: "We need to be careful with the parentheses. How about **v1∧**(**v2∧v3**)? The vector product **v2∧v3** is itself a vector and as such can form a multiplicand with a third vector. So it is a legitimate addition to Newton's table. But what does it equal?

Newton: "It's not obvious to me.

Breton: "Nor to me. Let me try to analyze the question. We know
(**v2∧v3**) = qv2∗qv3∗sin(angle(2,3))∗**un**(2,3)



v1∧(**v2∧v3**) = qv1∗qv2∗qv3∗sin(angle(2,3))
∗**uv1∧un**(2,3)

So the vector **uv1∧un**(2,3) is orthogonal to both **uv1** and **un**(2,3) while **un**(2,3) is itself orthogonal to **v2** and **v3**. Therefore since **v1∧un**(2,3) must lie in the plane generated by **v2** and **v3**

v1∧(**v2∧v3**) = a∗**v2** + b∗**v3**

for some a and b.

Newton: "Not bad. So we need only determine two scalar quantities, a and b.

Einstein: "The factor, qv1∗qv2∗qv3, shows we are dealing with some kind of volume which is not a scalar like **v1•**(**v2∧v3**) but a vector—a vector‒volume. Is this the same as a volume of vectors?

Breton: "You have an inquisitive mind Einstein. Like any vector a vectorial volume has a magnitude and a direction. We have discovered that the vectorial volume **v1∧**(**v2∧v3**) can be decomposed into two other vectorial volumes, one in the direction **uv2** and another in the direction **uv3**. But let us put your question aside for now as a distraction. Right now we are trying to obtain an equation for Newton's table.

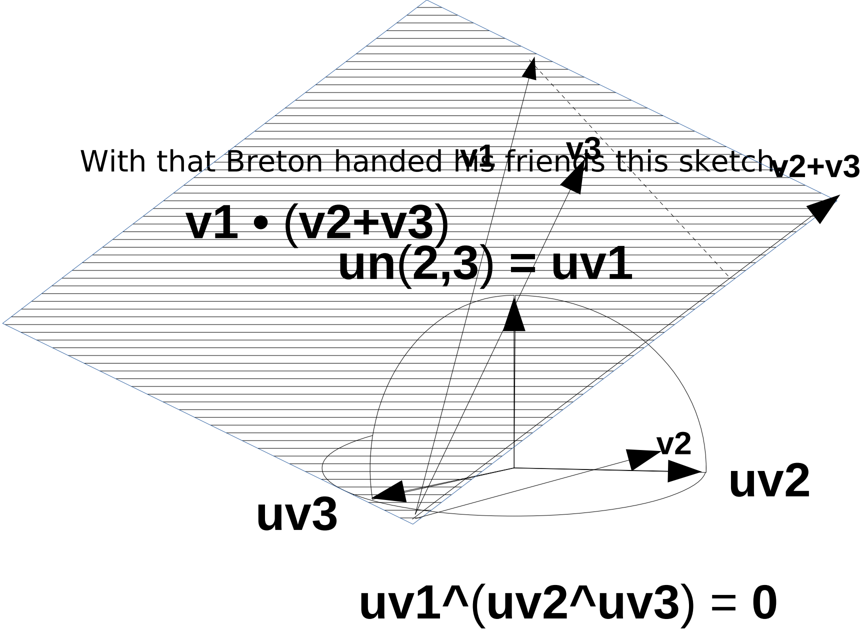
Newton: "Which we have reduced to determining two scalars, *a* and *b*. It seems to me that both *a* and *b* must somehow involve **v1**.

Breton: "You have good instincts, Newton. Furthermore, *a* and *b* must both be scalar “areas”.

Newton: "How do we proceed?

Breton: "Let's start with some examples which may show us the way. The path ahead looks rough, hard to cut through. And let's just consider the directions because we know the factor qv1∗qv2∗qv3 will finally apply to the full volume. So first look at a cube where the edges and their directions coincide the the vectors. Here look at this diagram.

With that Breton handed his friends this sketch.



Breton: "You are looking at a corner of the cube whether from the inside of from the outside makes no difference. For this geometry .**v2** and **v3** are orthogonal to each other and **v1** is orthogonal to both. For this example,

$$(\mathbf{uv2} \wedge \mathbf{uv3}) = \mathbf{uv1})$$

since sin(angle(2,3)) = 1.
Then

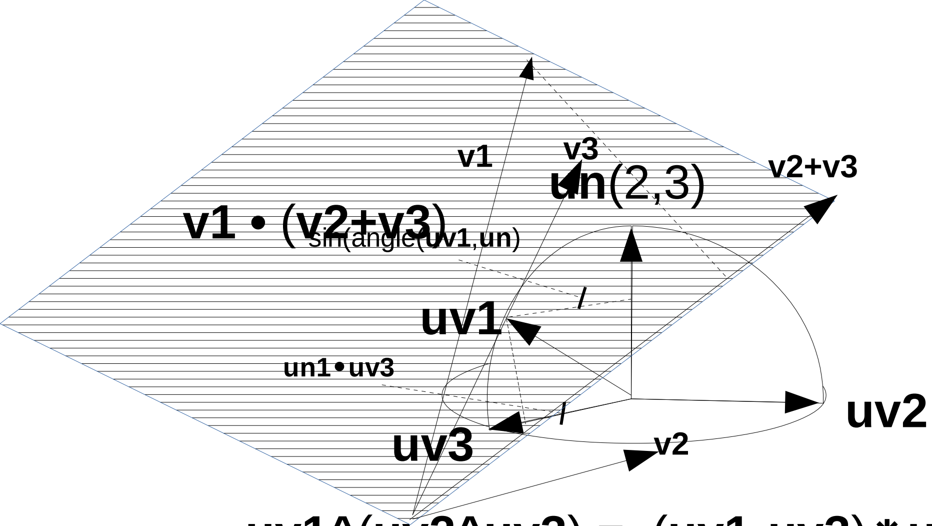
$$\mathbf{uv1} \wedge (\mathbf{uv2} \wedge \mathbf{uv3}) = \mathbf{uv1} \wedge \mathbf{uv1} = \mathbf{0}$$

Newton: "Not much learned from this example. The result would hold for any rectangular parallelepiped. Another example?

Breton: "Let's incline **uv1** in the **uv3** direction Then again

$$(\mathbf{uv2} \wedge \mathbf{uv3}) = \mathbf{un}(2,3)$$

as in this sketch.



uv1^(uv2^uv3) = (uv1•uv3)∗uv2

Now

uv1∧(**un**(2,3)) = sin(angle(1,un)∗**uv2**.

Notice sin(angle(1,un) = cos(angle(1,3)

Newton: "And cos(angle(1,3) = **uv1•uv3**. So for this example

uv1∧(**uv2∧uv3**) = (**uv1•uv3**)∗**uv2**.

Again this example could be expanded to parallelepipeds similarly inclined.

Breton: "We might have inclined **uv1** in the **uv2** direction and so obtained

uv1∧(**uv2∧uv3**) = (**uv1•uv2**)∗**uv3**

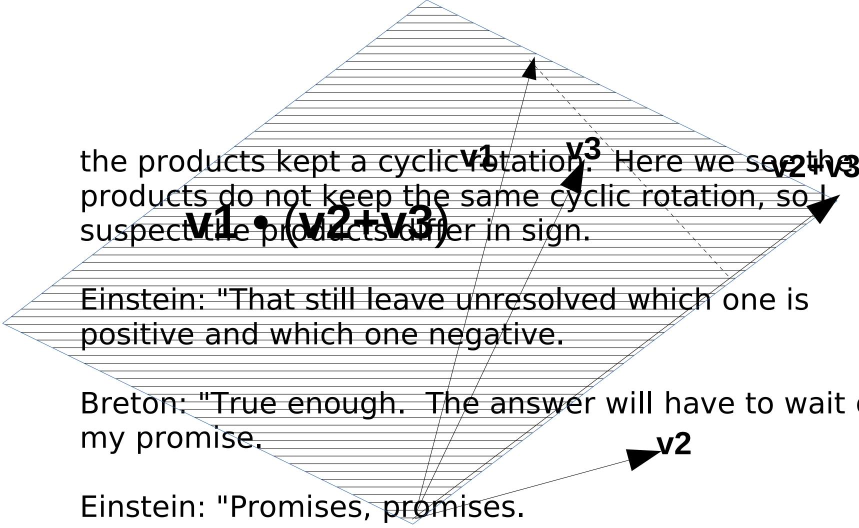
Einstein: "Wait a minute. We have not decided the direction of the cross product. It could be positive or negative. Which one is it here?

Breton: "Very little gets by you Einstein. I have promised resolution of this problem, but as of now, I have not delivered on that promise. So the two products,

(**uv1•uv3**)∗**uv2** and (**uv1•uv2**)∗**uv3**

might both be positive, or both negative, or one positive and the other negative.

Newton: "I suspect we can sharpen the result a little. Recall we found previously that the cross products could be organized into two groups by sign. Within each group



the products kept a cyclic rotation. Here we see the products do not keep the same cyclic rotation, so I suspect the products differ in sign.

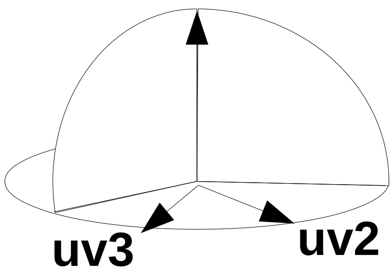
Einstein: "That still leave unresolved which one is positive and which one negative.

Breton: "True enough. The answer will have to wait on my promise.

Einstein: "Promises, promises.

Breton: "One or the other is true, so we have indeed advanced toward a solution even if we not attained it completely. For now let's continue our search for a comprehensive solution. Suppose **v2** and **v3** and not orthogonal, while **v1** is orthogonal to both.

un(2,3) = **v1**



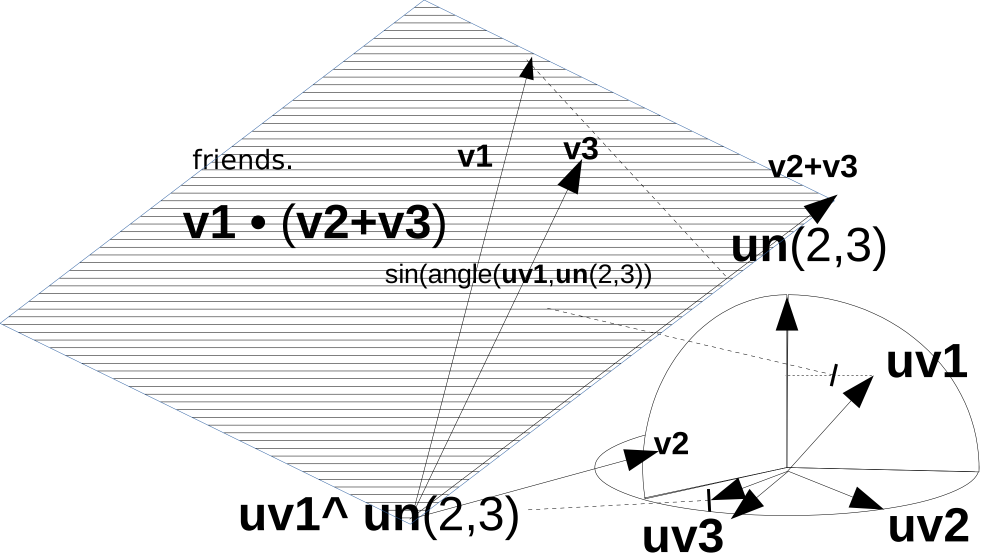
uv1^(uv2^uv3) = 0

not much different from the fully orthogonal case since **uv2∧uv3** = sin(angle(2,3)⁎**uv1** and **uv1∧uv1** = **0**

Breton: "Good! And if **uv1** is inclined toward **uv2** or **uv3** we get the same answer as as before. So of all the possible case we might explore, only one is left—the case where **uv1** is inclined arbitrarily.

With that Breton handed the following illustration to his

friends.



Breton: "The vector **uv1∧un**(2,3) is orthogonal to both **uv1** and **un**.

Einstein: " But **uv1∧un**(2,3) does not equal **uv1∧**(**uv2∧uv3**)!

Breton: "True enough, but close. As we have seen

uv1∧(**uv2∧uv3**) = sin(angle(2,3))∗**uv1∧un**(2,3)

so we seek a vector parallel to the one illustrated.

Einstein: "In fact,

uv1∧(**uv2∧uv3**)

= sin(angle(2,3))∗sin(angle(**uv1**,**un**(2,3))

∗**un**(**uv1**,**un**(2,3))

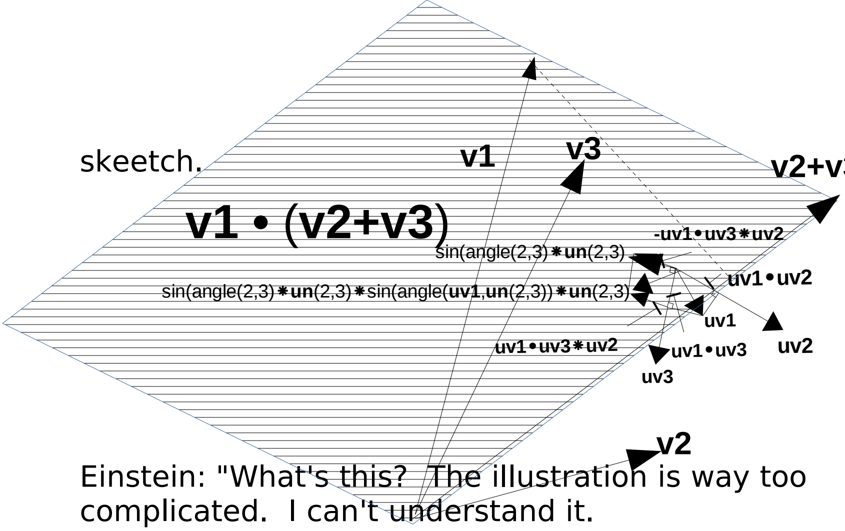
Newton: " So we have the right direction and length. What more do we want?

Breton: "You have noted the result is a vector in the plane defined by **uv2** and **uv3**. So can we express the result as a vector in that plane?

Einstein: "Breton draw a diagram showing all the vectors!

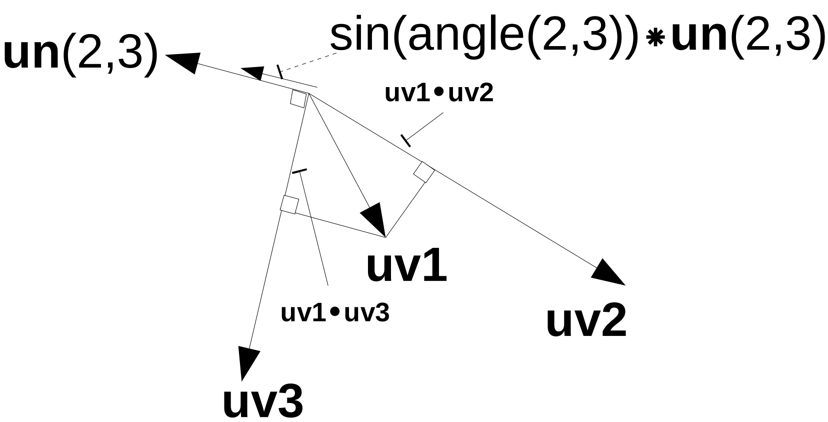
Breton: "All right, but it could be complicated. Within a few minutes Breton presented his friends the following

skeetch.



Einstein: "What's this? The illustration is way too complicated. I can't understand it.

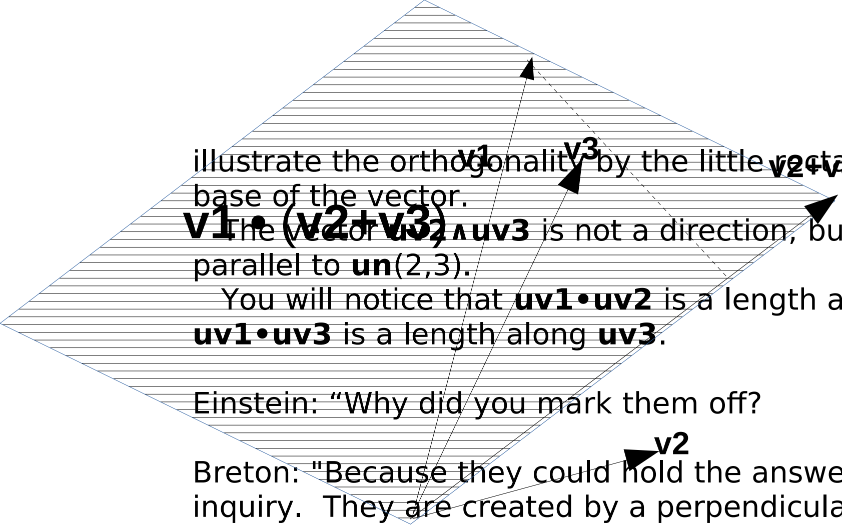
Breton: "As I suspected. Let me build it up slowly. With that he presented his friends the following illustration.



Breton: "We start with three directions, **uv1**, **uv2**, and **uv3**. Two of the vectors, **uv2** and **uv3**, lie in a plane, the plane of the illustration. The other vector, **uv1**, inclines from the plane. All three vectors have the same unit magnitude, so you have to imagine the position of **uv1**.

Newton: "They all are are radii of a sphere which we can imagine like a bubble enclosing the illustration.

Breton: "Exactly. Then **un**(2,3) is another such radius which is orthogonal to both **uv2** and **uv3**. I tried to



illustrate the orthogonality by the little rectangle at the base of the vector.

The vector **uv2∧uv3** is not a direction, but a vector parallel to **un**(2,3).

You will notice that **uv1•uv2** is a length along **uv2** and **uv1•uv3** is a length along **uv3**.

Einstein: “Why did you mark them off?

Breton: "Because they could hold the answer to our inquiry. They are created by a perpendicular from the tip of **uv1** to the lines of both **uv2** and **uv3**. They create slanted triangles.

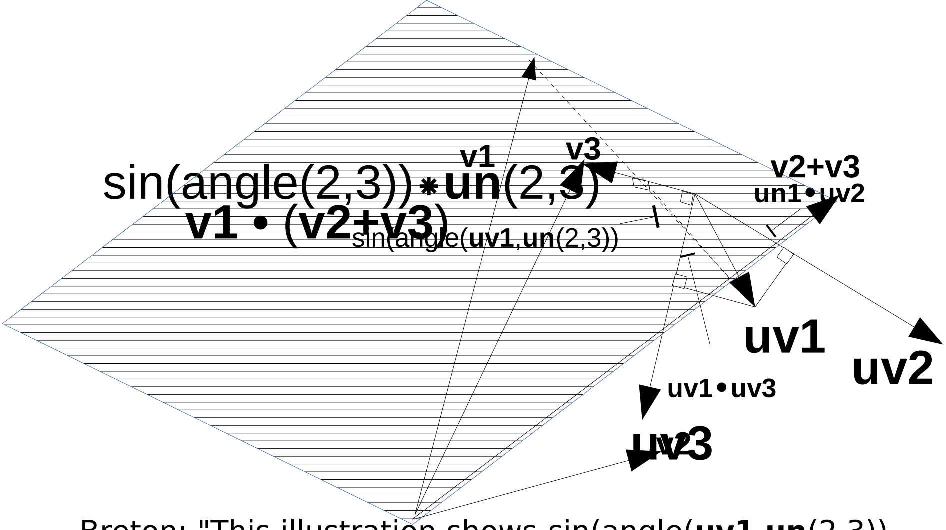
Recall the case where **uv1** was in the plane of **un**(2,3) and **uv2**? Then **uv1∧ un**(2,3) = **uv1•uv2∗uv3.** So I thought to mark them on the illustration first to show how they corroborate our earlier conclusions and then to open a path which might lead to an answer to our inquiry.

Einstein: "Then if **uv1** lies in the plane of **un**(2,3) and **uv3** then **uv1∧ un**(2,3) = –**uv1•uv3∗uv2** as I expected.

Breton: "That is still an undecided question. But you do observe correctly that the rotation of **uv1∧ un**(2,3) is opposite for **uv3** from the **uv2**.

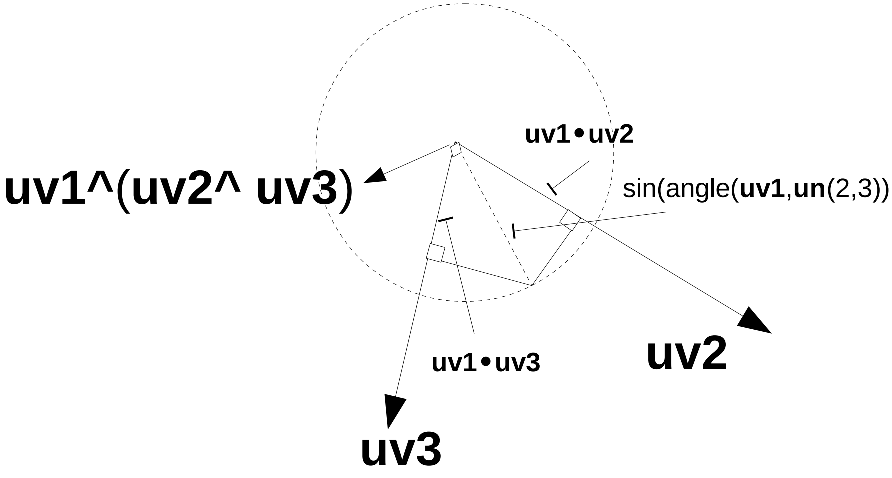
Newton: "All right the diagram illustrates all of the results obtained so far. Now return to the general case.

With Newton's words, Breton quickly sketched the following illustration.



Breton: "This illustration shows sin(angle(**uv1**,**un**(2,3)), the projection of which on the plane of **uv2** and **uv3** would lie directly under the **uv1** vector. The vector **uv1∧(**sin(angle(2,3))⁎**un**(2,3)) lies in the plane of **uv2** and **uv3** since it is orthogonal to **un**(2,3).

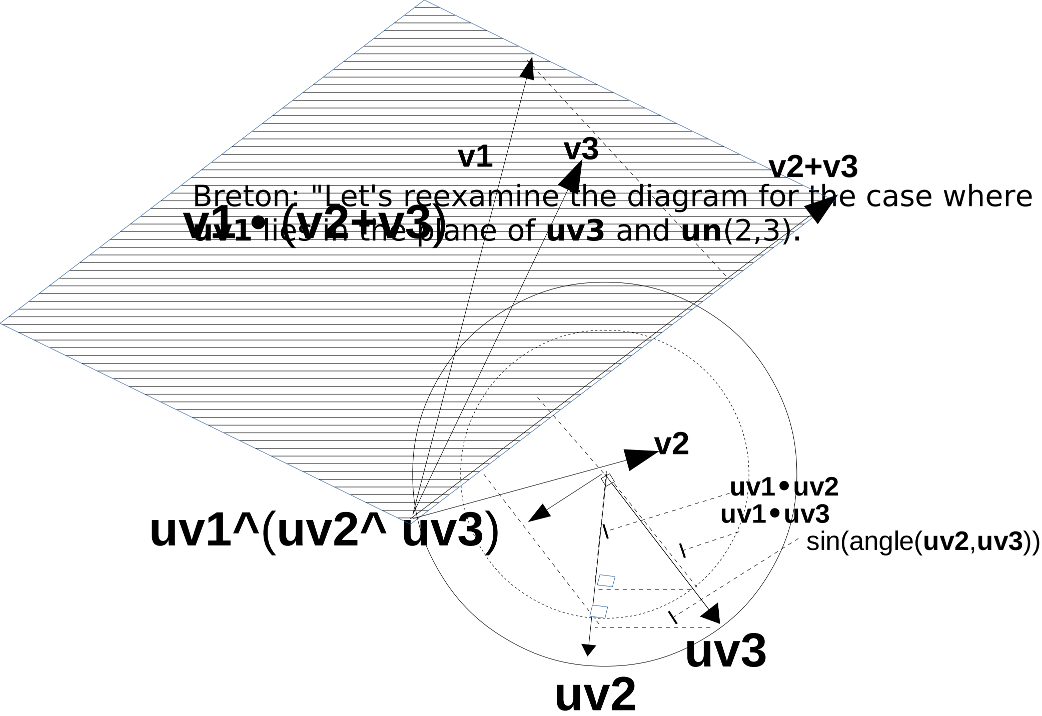
So now we can focus on the plane of **uv2** and **uv3** as in this next illustration.



Newton: "We are no longer referenced to a unit sphere, but to some smaller sphere inside it.

Einstein: "Just how are the results expressed in terms of **uv2** and **uv3**?

Breton: "Let's reexamine the diagram for the case where **uv1** lies in the plane of **uv3** and **un**(2,3).



And even more particularly, the case where **uv1** = **uv3**
In this case,

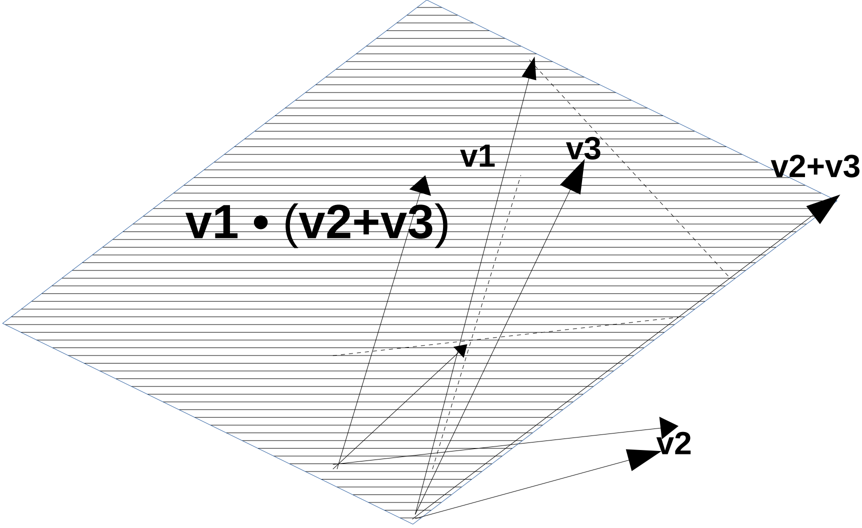
uv1•uv3 = 1
uv1•uv2 = sin(angle(**uv2**,**uv3**))
sin(angle(**uv1**,**un**(2,3)) = 1

Then
uv3∧(**uv2∧uv3**)
= sin(angle(**uv2**,**uv3**))
∗sin(angle(**uv3**,**un**(2,3))∗**un**(**uv3**,**un**(2,3))
= **uv3•uv2**∗1∗**un**(**uv3**,**un**(2,3))

Einstein: "How does this relate to **uv2** and **uv3**?

Breton: "Remember Newton's rhomboid? Our problem has been reduced to one where we know the sum of two vectors each of which we know the direction, but not the magnitude.

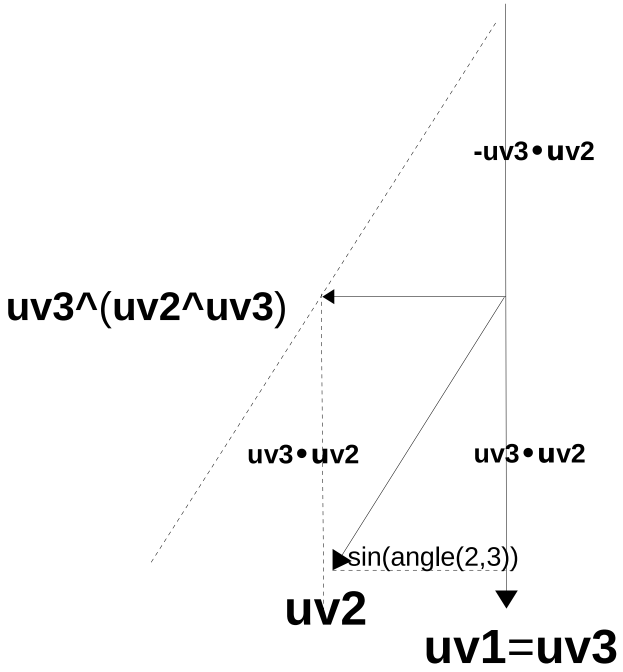
With that Breton sketched the following diagram for his friends.



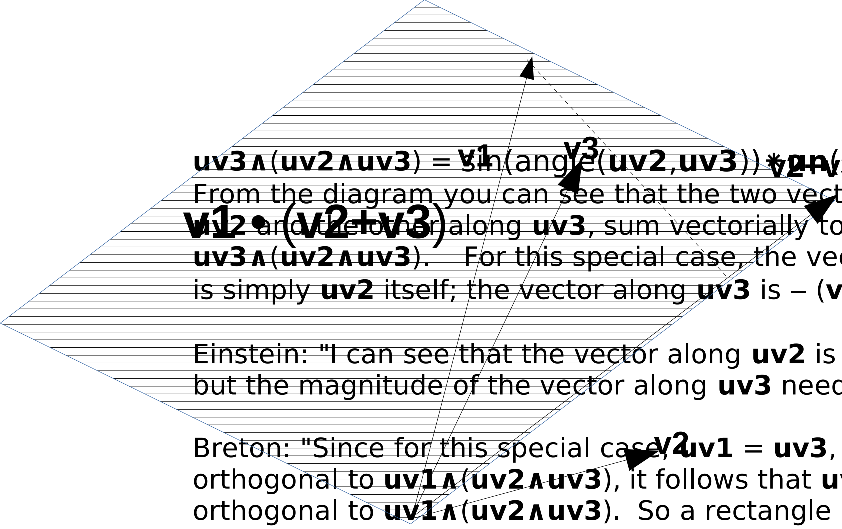
Breton: "See how the unknown magnitudes can become known by constructing parallel lines to the two vectors.

Einstein: " Show us how this solves our problem.

So Breton quickly produced the following sketch.



Breton: "We already know from above



uv3∧(**uv2∧uv3**) = sin(angle(**uv2**,**uv3**))∗**un**(**uv3**,**un**(2,3)).
From the diagram you can see that the two vectors, one along **uv2** and the other along **uv3**, sum vectorially to produce **uv3∧**(**uv2∧uv3**). For this special case, the vector along **uv2** is simply **uv2** itself; the vector along **uv3** is − (**v1• v2**)∗**v3**.

Einstein: "I can see that the vector along **uv2** is simply **uv2**, but the magnitude of the vector along **uv3** needs explanation.

Breton: "Since for this special case, **uv1** = **uv3**, and **uv1** is orthogonal to **uv1∧**(**uv2∧uv3**), it follows that **uv3** is also orthogonal to **uv1∧**(**uv2∧uv3**). So a rectangle is formed the length of whose sides are

uv3•uv2 and

sin(angle(**uv2**,**uv3**)).

The vector **uv2** splits the rectangle into two right triangles. The triangle on the top of the rectangle matches exactly the lower of these triangles.

Newton: "I can see that the the magnitude of **uv3∧**(**uv2∧uv3**) is simply sin(angle(**uv2**,**uv3**)).

Breton: "So we have shown for this special case

uv3∧(**uv2∧uv3**) = **uv2** − (**uv3•uv2**)∗**uv3**

= sin(angle(**uv2**,**uv3**))

Einstein: "As I expected the summation involves a negative vector.

Breton: "The question of the signs of vector products will be settled anon.

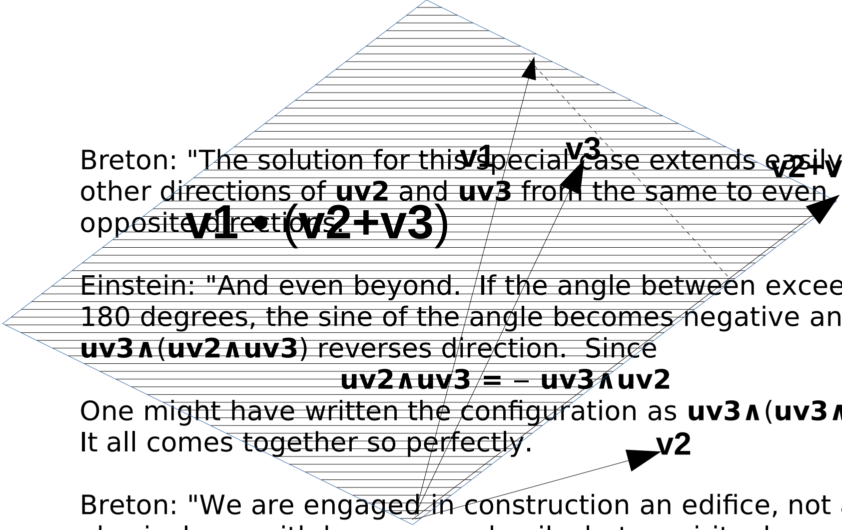
Einstein: "The diagram and the algebraic result mesh nicely. Look as the angle between **uv2** and **uv3** decreases, the diagram shows **uv3∧**(**uv2∧uv3**) becoming **0** and then likewise

uv3 − (**uv3•uv3**)∗**uv3** = **0.**

Newton: "And if the angle between the vectors grows to ninety degrees, **uv3∧**(**uv2∧uv3**) becomes **uv2** and likewise

uv2 − (**uv3•uv2**)∗**uv3 = uv2**

since **uv3•uv2** = **0**.



Breton: "The solution for this special case extends easily to other directions of **uv2** and **uv3** from the same to even opposite directions.

Einstein: "And even beyond. If the angle between exceeds 180 degrees, the sine of the angle becomes negative and **uv3∧**(**uv2∧uv3**) reverses direction. Since

uv2∧uv3 = – uv3∧uv2

One might have written the configuration as **uv3∧**(**uv3∧uv2**). It all comes together so perfectly.

Breton: "We are engaged in construction an edifice, not a physical one with hammer and nails, but a spiritual one with ideas. It's beginning to look beautiful.

Einstein, succumbing finally to the metaphor: "Let's continue the construction!

Breton: "Next then let's keep **uv1** in the plane of **uv3** and **un**(2,3) but not necessarily equal to **uv3**.

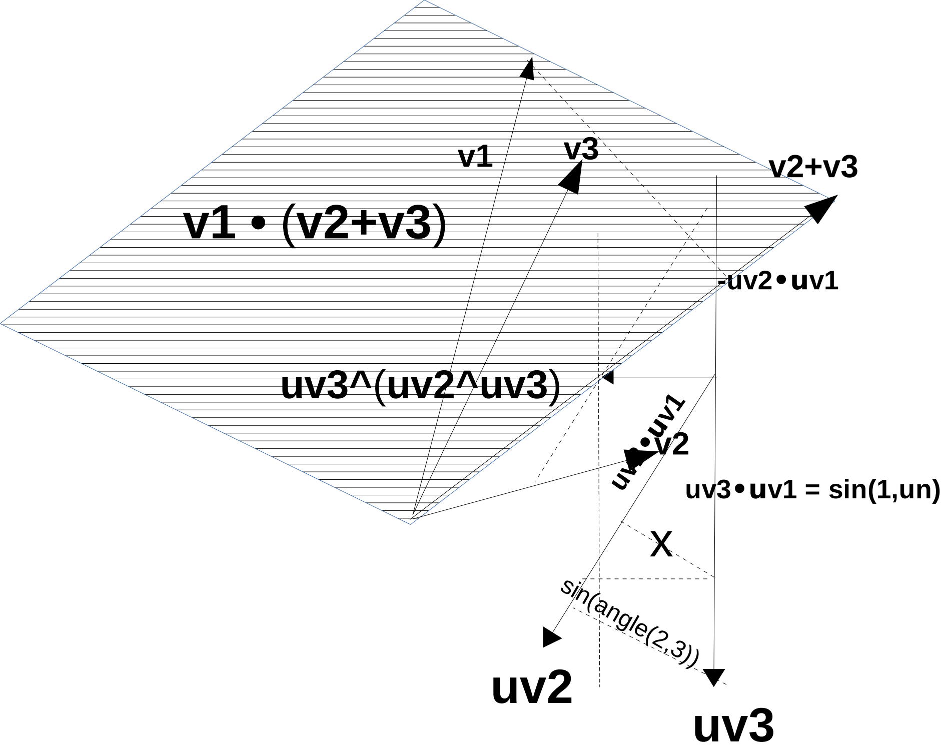
Newton: "Then **uv1∧**(**uv2∧uv3**) remains orthogonal to **uv1** and

uv1∧(**uv2∧uv3**) **=** sin(angle(2,3))
  *****sin(angle(**uv1**,**un**(2,3))
  *******un**(**uv1**,**un**(2,3)).

Einstein: "Then **uv1•uv3** no longer equals one.

Breton draw us a new diagram.

Breton: "With that Breton quickly produced the following sketch

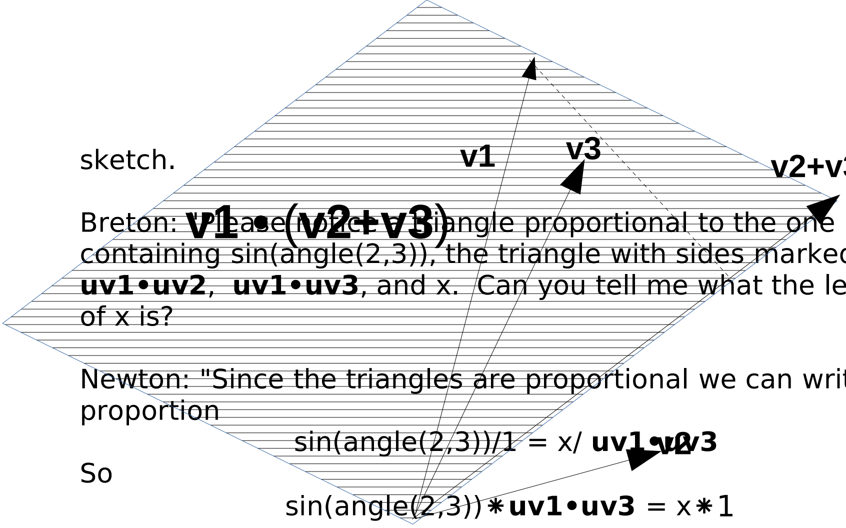


Breton: "The sketch looks down on the (**uv2**,**uv3**) plane. You will have to imagine the **un**(2,3) vector sticking straight up towards you. The **uv1** vector also sticks up from the plane but at an angle and lies in the plane of **uv3** and **un**(2,3). In addition to the planar vectors, I have also marked some magnitudes as projections of **uv1** on the plane: **uv1•uv3** which projects directly downwards on **uv3** and **uv1•uv2** which projects sidewards towards **uv2**.

Einstein: "Why?

Breton: "A little patience please. We know
uv1^(**uv2^uv3**) = sin(angle(2,3))*sin(angle(**uv1**,**un**(2,3))
***un**(**uv1**,**un**(2,3))
The sin(angle(2,3)) is marked on the sketch. As you can see it forms part of a right triangle whose hypotenuse is **uv3** with a magnitude equal to one.
The projections of **uv1** on **uv2** and **uv3** are their inner products and so marked on the sketch.

Newton: "They give a idea of the location of **uv1** up from the



sketch.

Breton: "Please notice a triangle proportional to the one containing sin(angle(2,3)), the triangle with sides marked **uv1•uv2**, **uv1•uv3**, and x. Can you tell me what the length of x is?

Newton: "Since the triangles are proportional we can write this proportion

sin(angle(2,3))/1 = x/ **uv1•uv3**

So

sin(angle(2,3))∗**uv1•uv3** = x∗1

that is,

x= sin(angle(2,3))∗**uv1•uv3**

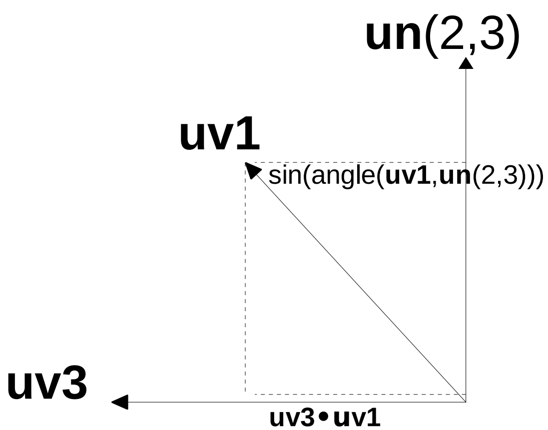
Einstein, dismissibely: "That much is obvious.

Breton: "Notice again that

uv1•uv3 = sin(angle(**uv1**,**un**(2,3)).

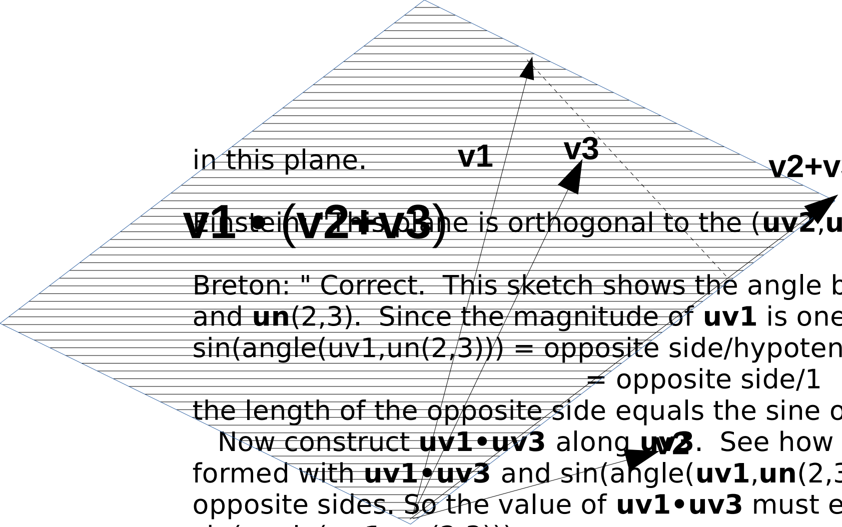
Einstein: "Show me.

Acceding to Einstein's request, Breton quickly sketched the following.



Breton: "In this sketch we are looking at the plane (**uv3**,**un**(2,3)). For the case we are considering **uv1** also lies

in this plane.



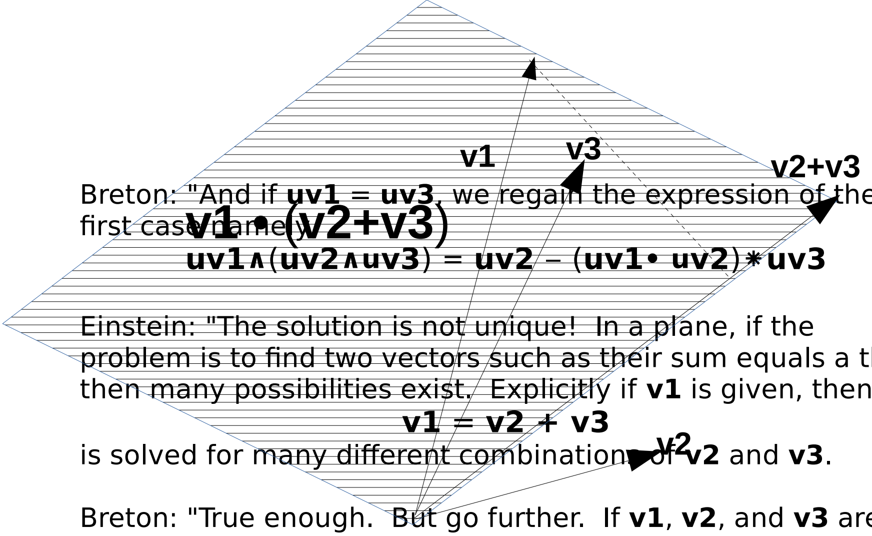
Einstein: "This plane is orthogonal to the (**uv2**,**uv3**) plane.

Breton: " Correct. This sketch shows the angle between **uv1** and **un**(2,3). Since the magnitude of **uv1** is one,
sin(angle(uv1,un(2,3))) = opposite side/hypotenuse
= opposite side/1
the length of the opposite side equals the sine of the angle.
Now construct **uv1•uv3** along **uv3**. See how .a rectangle is formed with **uv1•uv3** and sin(angle(**uv1**,**un**(2,3))) forming opposite sides. So the value of **uv1•uv3** must equal that of sin(angle(**uv1**,**un**(2,3))).

Newton, reflecting Einstein previous dismissive remark: "That much is obvious. The sides of the rectangle are
sin(angle(**uv1**,**un**(2,3))) = **uv1•uv3**
and
cos(angle(**uv1**,**un**(2,3))) = **uv1•un**(2,3)

Breton: "Exactly. So now we have established that the value of x in the prior sketch equals
sin(angle(2,3))∗sin(angle(**uv1**,**un**(2,3))
which is just the magnitude of **uv1^**(**uv2^uv3**).
Now let us return to the prior sketch and focus on the right triangle formed by the sides
x
uv1•uv3
and
uv1•uv2
Now construct a similar right triangle above this with the sides
sin(angle(2,3))∗sin(angle(**uv1**,**un**(2,3))
along **uv1^**(**uv2^uv3**).
−**uv1•uv2**
along **uv3**
and a remaining side
along **uv2**.
The length of the remaining side must be **uv1•uv3**. I have marked this constructed triangle in the sketch.

Newton: "So we have just proven
uv1∧(**uv2∧uv3**) = (**uv1•uv3**)∗**uv2** − (**uv1• uv2**)∗**uv3**



Breton: "And if **uv1** = **uv3**, we regain the expression of the first case, namely

$$\mathbf{uv1} \wedge (\mathbf{uv2} \wedge \mathbf{uv3}) = \mathbf{uv2} - (\mathbf{uv1} \bullet \mathbf{uv2}) * \mathbf{uv3}$$

Einstein: "The solution is not unique! In a plane, if the problem is to find two vectors such as their sum equals a third, then many possibilities exist. Explicitly if **v1** is given, then

$$\mathbf{v1} = \mathbf{v2} + \mathbf{v3}$$

is solved for many different combinations of **v2** and **v3**.

Breton: "True enough. But go further. If **v1**, **v2**, and **v3** are all given, then no solution may exist. Further if only **v1** and **v2** are given, then a unique **v3** is determined as

$$\mathbf{v3} = \mathbf{v1} - \mathbf{v2}$$

But the task we set ourselves is different from these. We are given **v1**, **v2**, and **v3**, and we seek a solution, if any, to

$$\mathbf{v1} = q2 * \mathbf{uv2} + q3 * \mathbf{uv3}$$

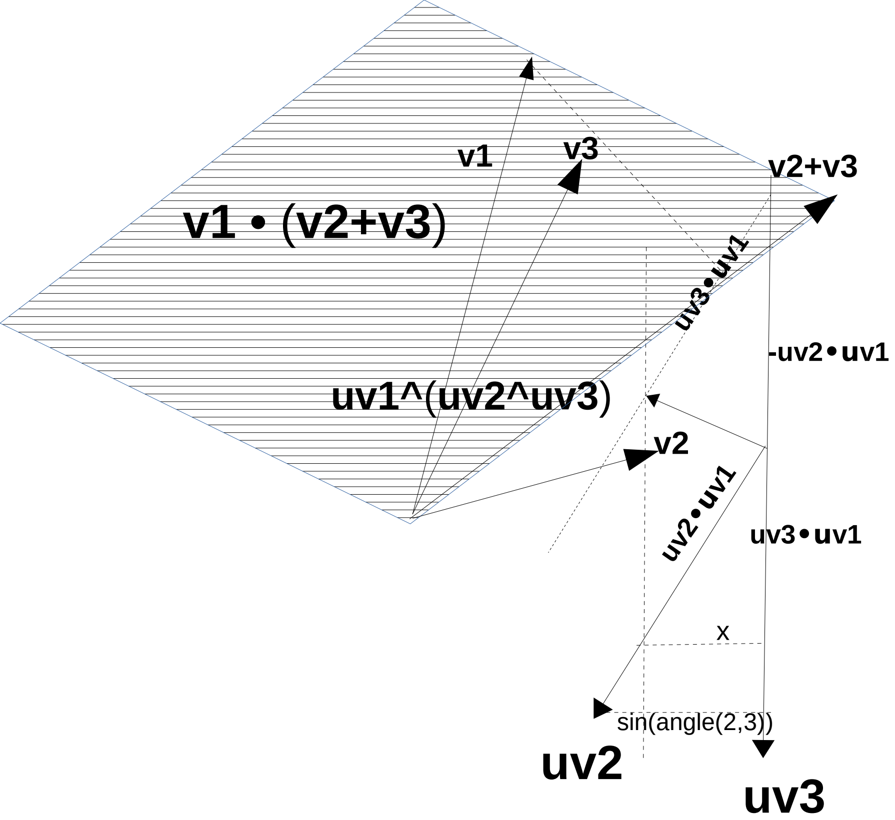
where q2 and q3 are unknown. We have found unique solutions for this problem.

Einstein: "What if **uv1** had been aligned with **uv2**?

Breton: "Let us investigate **uv1** in the plane of **uv2** and **un**(2,3). Then our results would include **uv1** = **uv2.**

Einstein: "Yes, that would do.

Breton: "Then consider the following sketch.



Breton: "Now **uv1∧**(**uv2∧uv3**) is orthogonal to **uv2** rather than **uv3** as before. The sin(angle(2,3)) is also represented differently. Again we ask for the magnitude of the line marked x.

Newton: "Again we see proportional triangle, so we can write

sin(angle(2,3))/1 = x/ **uv1•uv2**

So

x= sin(angle(2,3))∗**uv1•uv2**

Einstein: "That much is obvious. And now

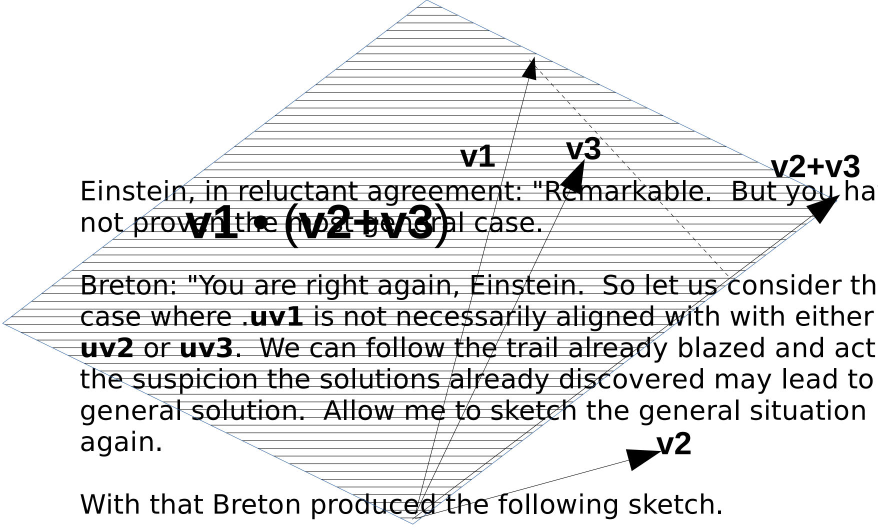
uv1•uv2 = sin(angle(**uv1**,**un**(2,3)).

So again the magnitude of x just equals the magnitude of **uv1∧**(**uv2∧uv3**).

Breton: "Can you see that the triangle with sides x, **uv1•uv2**, and **uv1•uv3** matches the triangle with sides **uv1∧**(**uv2∧uv3**), −**uv1•uv2**, and **uv1•uv3** exactly?

Newton: "So again,

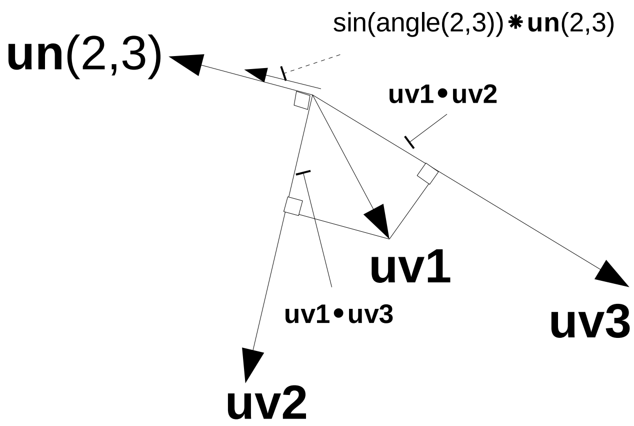
uv1∧(**uv2∧uv3**) = (**uv1•uv3**)∗**uv2** – (**uv1• uv2**)∗**uv3**



Einstein, in reluctant agreement: "Remarkable. But you have not proven the most general case.

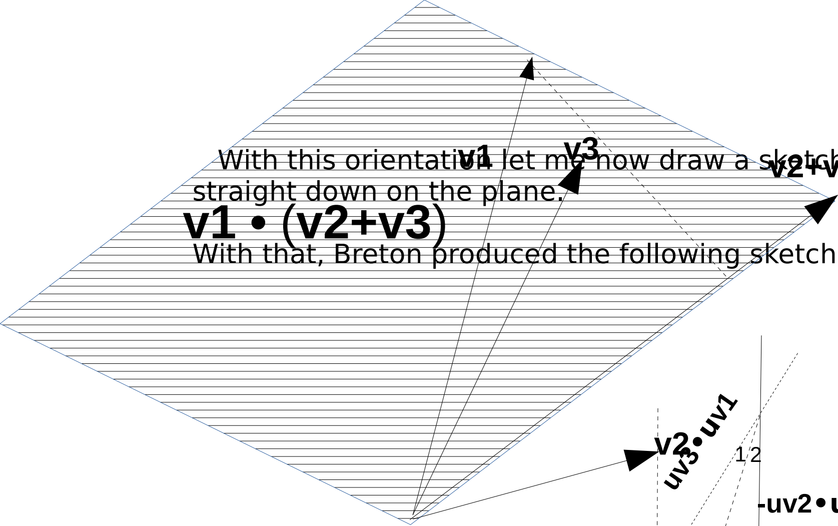
Breton: "You are right again, Einstein. So let us consider the case where .**uv1** is not necessarily aligned with with either **uv2** or **uv3**. We can follow the trail already blazed and act on the suspicion the solutions already discovered may lead to the general solution. Allow me to sketch the general situation again.

With that Breton produced the following sketch.

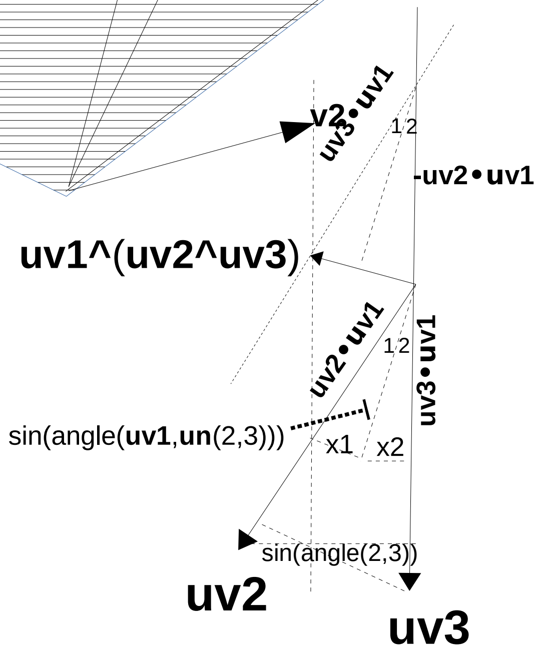


Breton: "The sketch imagines you are looking at the (**uv2**,**uv3**) plane from an angle rather than straight down. All of the directions can be shown, remembering that each direction has a length equal to one. The directions **uv2** and **uv3** determine **un**(2,3) and the sine of the angle between them determines sin(angle(2,3))∗**un**(2,3). The direction **uv1** sticks up from the plane and determines **uv1•uv2** and **uv1•uv3**. Again sin(angle(**uv1**,**un**(2,3))) equals the projection of **uv1** onto the plane directly below.

With this orientation let me now draw a sketch looking straight down on the plane.



With that, Breton produced the following sketch.

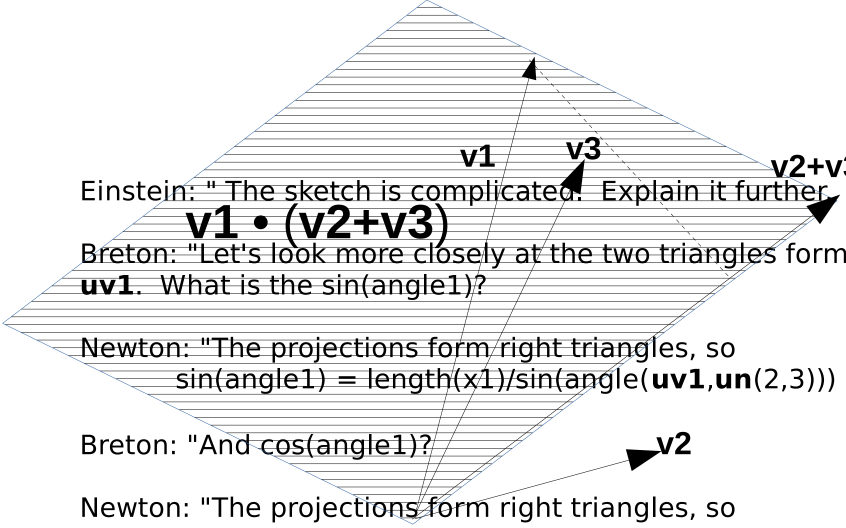


Breton: "For this case, looking down on the (**uv2**,**uv3**) plane, the magnitude of **uv1•uv3** and **uv1•uv2** are shown from orthogonal projections from **uv1** onto **uv3** and **uv2**.   The projection of **uv1** onto the plane provides the magnitude of sin(angle(**uv1**,**un**(2,3))). The line x1 parallels one depiction of sin(angle(2,3); x2 parallels the alternate depiction. Each connects the tip of the **uv1** projection to the projections onto **uv2** or **uv3**.
Sin(angle(2,3) equals the inner product of **uv3** and **uv2** matching the way sin(angle(2,3)) is shown.

In the general case **uv1∧**(**uv2∧uv3**) is orthogonal to **uv1**, but not necessarily to either **uv2** or **uv3**.

Note that angle(2,3) is equal to the sum of two angles, labeled 1 and 2 created by **uv1**.

Newton: "Now there are no proportional triangles.



Einstein: " The sketch is complicated. Explain it further.

Breton: "Let's look more closely at the two triangles formed by **uv1**. What is the sin(angle1)?

Newton: "The projections form right triangles, so
sin(angle1) = length(x1)/sin(angle(**uv1**,**un**(2,3)))

Breton: "And cos(angle1)?

Newton: "The projections form right triangles, so
cos(angle1) = **uv1•uv2**/sin(angle(**uv1**,**un**(2,3)))

Breton: "What is the sin(angle2)?

Newton: "Again we find a right triangle, so
sin(angle2) = length(x2)/sin(angle(**uv1**,**un**(2,3)))

Breton: "And cos(angle2)?

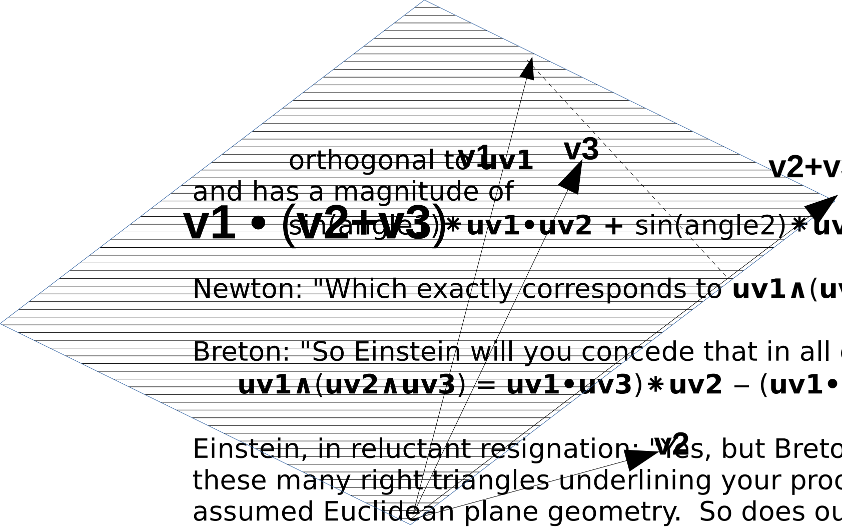
Newton: "That's easy.
cos(angle2) = **uv1•uv3**/sin(angle(**uv1**,**un**(2,3)))

Einstein: "And what has all this to do with your proof?

Breton: "Patience, my dear Einstein. Now consider the upper triangle formed by the sides labeled **uv1•uv2**, **uv1•uv3**, and **uv1∧**(**uv2∧uv3**). The upper angle is just angle(2,3) and so may be divided into our two angles, 1 and 2, as in the lower triangle. The dividing line is now orthogonal to **uv1∧**(**uv2∧uv3**).

Einstein: "So we can calculate
uv1•uv3)∗**uv2** − (**uv1•uv2**)∗**uv3**.

Breton: "Exactly. Start with
sin(angle1) = x1/**uv1•uv2**
sin(angle2) = x2/**uv1•uv3**
from which we can calculate
length(x1+x2) = sin(angle1)∗**uv1•uv2**
+ sin(angle2)∗**uv1•uv3**
Then
uv1•uv3)∗**uv2** − (**uv1•uv2**)∗**uv3** is a vector which is



orthogonal to **uv1**
and has a magnitude of
sin(angle1)∗**uv1•uv2 +** sin(angle2)∗**uv1•uv3**

Newton: "Which exactly corresponds to **uv1∧**(**uv2∧uv3**).

Breton: "So Einstein will you concede that in all cases
uv1∧(**uv2∧uv3**) = **uv1•uv3**)∗**uv2** − (**uv1•uv2**)∗**uv3**?

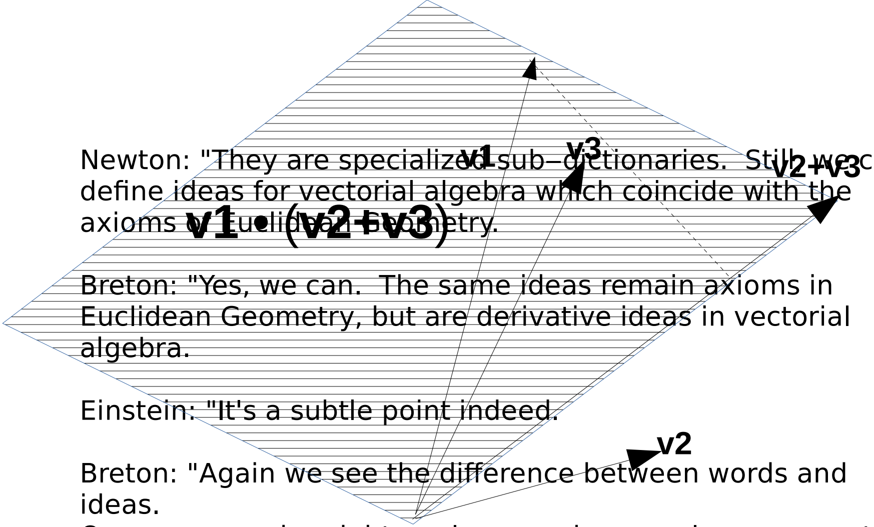
Einstein, in reluctant resignation: "Yes, but Breton with all these many right triangles underlining your proof, you have assumed Euclidean plane geometry. So does our vector algebra rest on Euclid's axioms?

Newton: "What an amazingly wonderful tribute to my illustrious ancestor.

Breton: "Not so. It is a point worth discussing. For his geometry Euclid includes an axiom called the **parallel postulate**, namely that parallel lines never meet. From this postulate he easily deduces that the interior angles of a triangle equal two right angles. This is the language of Euclidean geometry, a mathematical science.
The language of vectorial calculus is somewhat different. Parallel lines for this admittedly mathematical science are those which have the same direction. So even though the two sciences use the same word, *parallel*, the meaning of the word differs in each. So it is with many of the words we have been using like lines points, etc. which are axioms for Euclid, but in vectorial calculus are derivative ideas based on direction, magnitude, and the underlying field with its own calculus and topology.
So our words have been confusing two different dictionaries, somewhat like the confusion between the Mathematics and Theoretical Physics.

Newton: "Both dictionaries, the one for Eucldean geometry and the one for vectorial algebra are mathematical dictionaries.

Breton: "For this reason we are tempted to use the same words for different ideas. We saw in tp1.1 that any science cannot tolerate the ambiguity.



Newton: "They are specialized sub–dictionaries. Still we can define ideas for vectorial algebra which coincide with the axioms of Euclidean Geometry.

Breton: "Yes, we can. The same ideas remain axioms in Euclidean Geometry, but are derivative ideas in vectorial algebra.

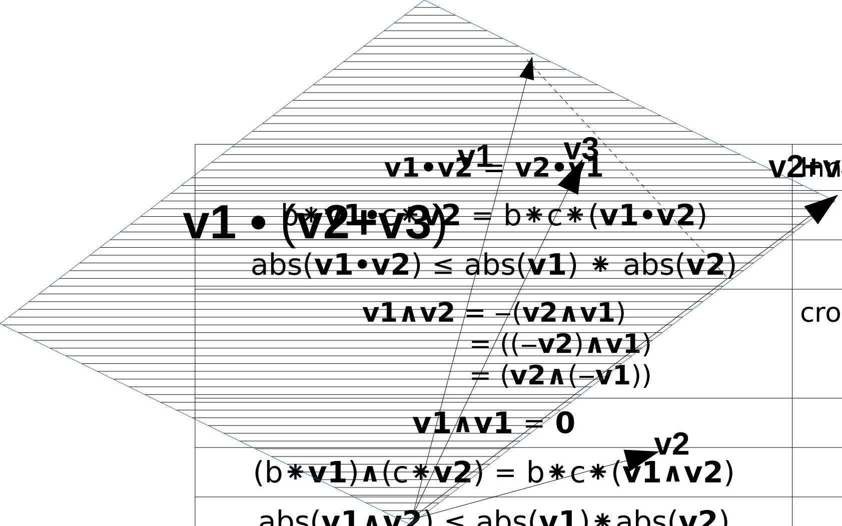
Einstein: "It's a subtle point indeed.

Breton: "Again we see the difference between words and ideas.
One can conceive right angles even in non–planar geometries. Moreover, even in a plane one can conceive geometries which are non–Euclidean. Although, when we *defined* triangles, we used the fact that in the Euclidean plane the sum of the angles in a triangle equals 180 degrees. These special definitions referred only to Euclidean plane triangles. Angles and triangles may be defined in a broader context. In this broader context, our results will still hold because the logic remains the same for these different triangles. In effect we used Euclidean Geometry to show results that apply generically. In the generic case, the axiomatic ideas of Euclidean Geometry would have to be defined in terms of vectorial algebra: direction, magnitude, and the scalar field. That said, Euclidean geometry does comport well with vector algebra.

Newton: "Previously we saw that vectorial proofs can facilitate geometrical proofs. In this latest proof we we see the reverse—geometrical proofs facilitate a vectorial conclusion. Vectorial algebra and Geometry march together like a groom and bride.

Breton: "Newton, would you update your table to includes our latest results?

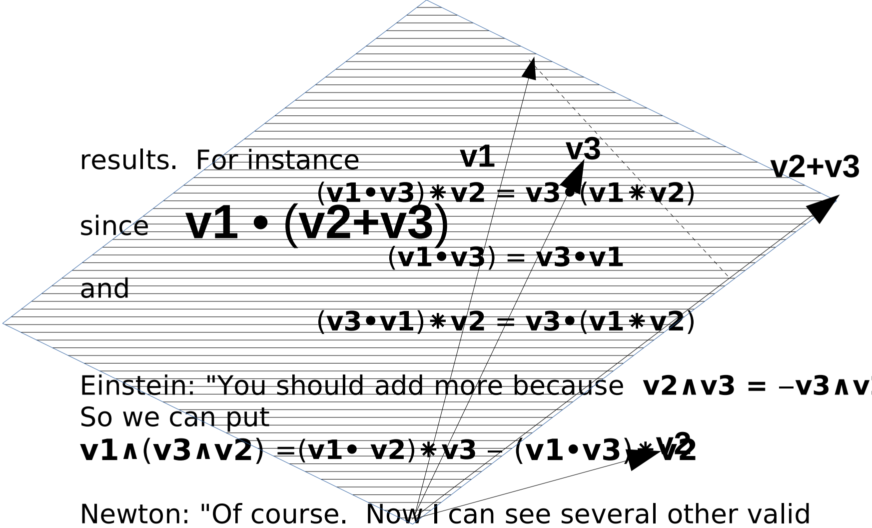
Axiomatic	**Comments**
v1+v2 = **v3**	closure
q∗**v1**= **v2**	Scalar multiply
v1+(**v2+v3**) = (**v1+v2**)**+v3**	association
Defined: two at a time	



v1•**v2** = **v2**•**v1**	inner product
(b∗**v1**)•(c∗**v2**) = b∗c∗(**v1**•**v2**)	
abs(**v1**•**v2**) ≤ abs(**v1**) ∗ abs(**v2**)	
v1∧**v2** = −(**v2**∧**v1**) = ((−**v2**)∧**v1**) = (**v2**∧(−**v1**))	cross product
v1∧**v1** = **0**	
(b∗**v1**)∧(c∗**v2**) = b∗c∗(**v1**∧**v2**)	
abs(**v1**∧**v2**) ≤ abs(**v1**)∗abs(**v2**)	
v1•(**v1**∧**v2**) = **v2**•(**v1**∧**v2**) = 0	
(b∗**v1**)∗(c∗**v2**) = b∗c∗(**v1**∗**v2**)	
(abs(**v1**)∗abs(**v2**))^2 = (abs(**v1**∧**v2**))^2 + (abs(**v1**•**v2**))^2	
Defined: three at at time	
v1•(**v2**+**v3**) = **v1**•**v2** + **v1**•**v3**	
v1∧(**v2**+**v3**) = **v1**∧**v2** + **v1**∧**v3**	
v1∗(**v2**+**v3**) = **v1**∗**v2** + **v1**∗**v3**	
v1•(**v2**∧**v3**) = **v2**•(**v3**∧**v1**) = **v3**•(**v1**∧**v2**) = (**v1**∧**v2**)•**v3** = (**v2**∧**v3**)•**v1** = (**v3**∧**v1**)•**v2**	Scalar triple product
v1•(**v2**∗**v3**) = (**v1**•**v2**)∗**v3**	transformation
v1∧(**v2**∧**v3**) = (**v1**•**v3**)∗**v2** − (**v1**• **v2**)∗**v3** = **v3**•(**v1**∗**v2**) − **v3**∗(**v1**•**v2**) = **v1**•(**v3**∗**v2** − **v2**∗**v3**)	Vector triple prduct

Einstein: "I see you have added some additional results.

Newton: "Yes. They are simply elaborations from earlier



results. For instance

$$(\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2} = \mathbf{v3}\bullet(\mathbf{v1}\ast\mathbf{v2})$$

since

$$(\mathbf{v1}\bullet\mathbf{v3}) = \mathbf{v3}\bullet\mathbf{v1}$$

and

$$(\mathbf{v3}\bullet\mathbf{v1})\ast\mathbf{v2} = \mathbf{v3}\bullet(\mathbf{v1}\ast\mathbf{v2})$$

Einstein: "You should add more because $\mathbf{v2}\wedge\mathbf{v3} = -\mathbf{v3}\wedge\mathbf{v2}$. So we can put
$\mathbf{v1}\wedge(\mathbf{v3}\wedge\mathbf{v2}) = (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3} - (\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2}$

Newton: "Of course. Now I can see several other valid combinations as well. Let me update the table to include them too.

After a few minutes Newton produced the additions to his table.

Defined: three at at time	
$(\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3}$ $= (\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2} - (\mathbf{v2}\bullet\mathbf{v3})\ast\mathbf{v1}$ $= \mathbf{v1}\bullet(\mathbf{v3}\ast\mathbf{v2})$ $- \mathbf{v1}\ast(\mathbf{v3}\bullet\mathbf{v2})$ $= \mathbf{v3}\bullet(\mathbf{v1}\ast\mathbf{v2} - \mathbf{v2}\ast\mathbf{v1})$	
$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) - (\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3}$ $= \mathbf{v2}\bullet(\mathbf{v3}\ast\mathbf{v1} - \mathbf{v1}\ast\mathbf{v3})$ $\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3})$ $+\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v1})$ $+ \mathbf{v3}\wedge(\mathbf{v1}\wedge\mathbf{v2}) = \mathbf{0}$	

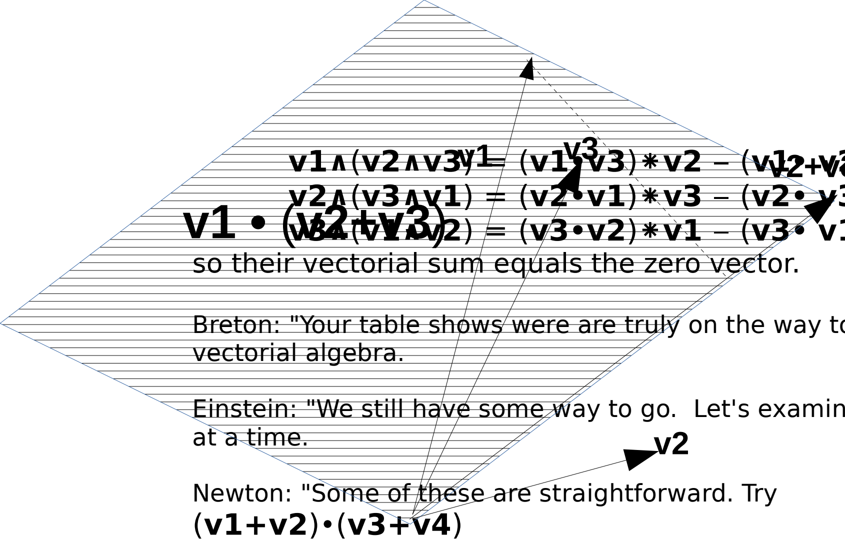
Newton: "The top entry uses your remark, Einstein, and relabels some of the vectors. The second entry finds an identity in certain of the differences. Now

$$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) = (\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2} - (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3}$$
$$(\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3} = (\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2} - (\mathbf{v2}\bullet\mathbf{v3})\ast\mathbf{v1}$$

so their difference

$$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) - (\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3} = -(\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3} + (\mathbf{v2}\bullet\mathbf{v3})\ast\mathbf{v1}$$
$$= \mathbf{v2}\bullet(\mathbf{v3}\ast\mathbf{v1} - \mathbf{v1}\ast\mathbf{v3})$$

The third item uses relabeling to discover a remarkable itdentity.



$$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) = (\mathbf{v1}\bullet\mathbf{v3})*\mathbf{v2} - (\mathbf{v1}\bullet\mathbf{v2})*\mathbf{v3}$$
$$\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v1}) = (\mathbf{v2}\bullet\mathbf{v1})*\mathbf{v3} - (\mathbf{v2}\bullet\mathbf{v3})*\mathbf{v1}$$
$$\mathbf{v3}\wedge(\mathbf{v1}\wedge\mathbf{v2}) = (\mathbf{v3}\bullet\mathbf{v2})*\mathbf{v1} - (\mathbf{v3}\bullet\mathbf{v1})*\mathbf{v2}$$

so their vectorial sum equals the zero vector.

Breton: "Your table shows were are truly on the way to developing a vectorial algebra.

Einstein: "We still have some way to go. Let's examine vectors four at a time.

Newton: "Some of these are straightforward. Try

$$(\mathbf{v1}+\mathbf{v2})\bullet(\mathbf{v3}+\mathbf{v4})$$
$$=\mathbf{v1}\bullet(\mathbf{v3}+\mathbf{v4}) + \mathbf{v2}\bullet(\mathbf{v3}+\mathbf{v4})$$
$$= \mathbf{v1}\bullet\mathbf{v3} + \mathbf{v1}\bullet\mathbf{v4} + \mathbf{v2}\bullet\mathbf{v3} + \mathbf{v2}\bullet\mathbf{v4}$$

Einstein: "Straightforward enough

Newton: "Then how about

$$(\mathbf{v1}+\mathbf{v2})\wedge(\mathbf{v3}+\mathbf{v4})$$
$$= \mathbf{v1}\wedge\mathbf{v3} + \mathbf{v1}\wedge\mathbf{v4} + \mathbf{v2}\wedge\mathbf{v3} + \mathbf{v2}\wedge\mathbf{v4}$$

and

$$(\mathbf{v1}+\mathbf{v2})*(\mathbf{v3}+\mathbf{v4})$$
$$= \mathbf{v1}*\mathbf{v3} + \mathbf{v1}*\mathbf{v4} + \mathbf{v2}*\mathbf{v3} + \mathbf{v2}*\mathbf{v4}$$

Einstein: "Yes, just use the same argument.

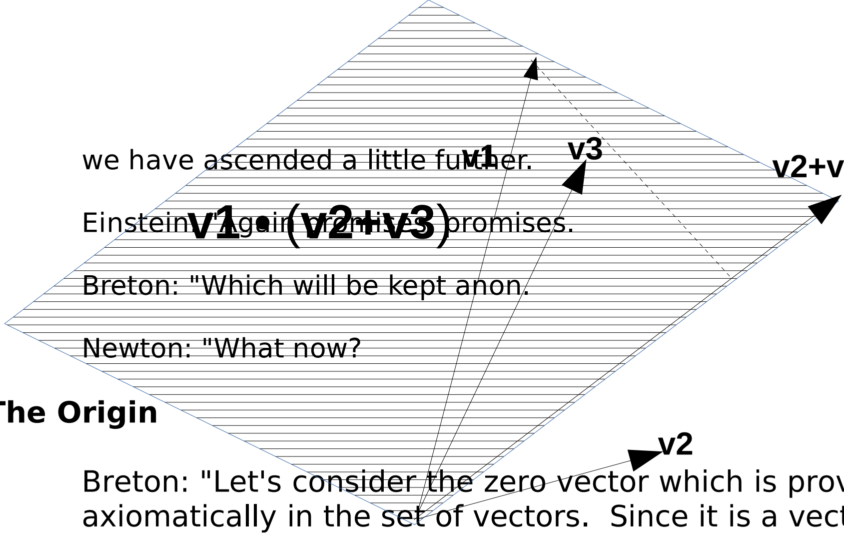
Newton: "Let's try something more difficult. Consider the scalar

$$(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4})$$

Einstein: "Not so difficult.

$$\begin{aligned}(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4}) &= q1*q2*\sin(\text{angle}1,2)\\ &\quad *q3*q4*\sin(\text{angle}3,4)\\ &\quad *\mathbf{un}(1,2)\bullet\mathbf{un}(3,4)\\ &= q1*q2*q3*q4\\ &\quad *\sin(\text{angle}1,2)*\sin(\text{angle}3,4)\\ &\quad *\cos(\mathbf{un}(1,2),\mathbf{un}(3,4))\end{aligned}$$

Breton: "Again we can ask for this relationship in terms of a purely vector equation, but our experience with triple products forewarns us of no small difficulties. May I suggest we abandon this part of the trail for now to take it up later when



we have ascended a little further.

Einstein: "Again, promises, promises.

Breton: "Which will be kept anon.

Newton: "What now?

The Origin

Breton: "Let's consider the zero vector which is provided axiomatically in the set of vectors. Since it is a vector, what is its magnitude?

Einstein: "That's easy. The zero vector has zero length.

Breton: "By which you must mean,

$$\mathbf{0} = 0 * u(\mathbf{0})$$

where zero in the set of vectors is not the same as zero in the underlying field of quotient numbers, Q.

Einstein: "Thank you for your precision.

Breton: "Now prove your assertion.

Einstein: "Of course, it's true.

Breton: "Which is merely your assertion. Why can't the zero vector have some other magnitude?

N,Why not?

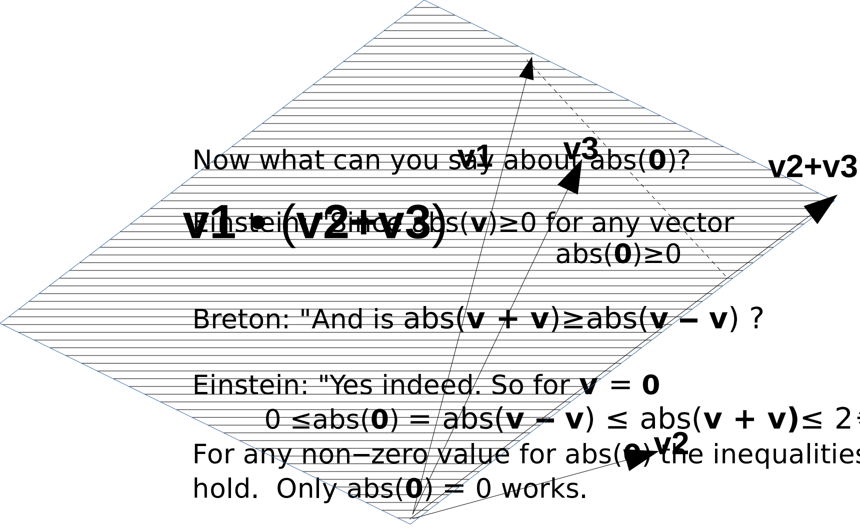
Einstein: "All right, let me try to prove something I already know is true. Where do I start?

Breton: "You might try the axioms.

Einstein: "Of course, the axioms are taken as true. Where shall we start?

Breton: "For any vector

$$1 * \mathbf{v} + 1 * \mathbf{v} = (1+1) * \mathbf{v} = 2 * \mathbf{v}$$
$$1 * \mathbf{v} - 1 * \mathbf{v} = (1-1) * \mathbf{v} = 0 * \mathbf{v}$$



Now what can you say about abs(**0**)?

Einstein: "Since abs(**v**)≥0 for any vector

$$\mathrm{abs}(\mathbf{0})\geq 0$$

Breton: "And is abs(**v + v**)≥abs(**v − v**) ?

Einstein: "Yes indeed. So for **v** = **0**

$$0 \leq \mathrm{abs}(\mathbf{0}) = \mathrm{abs}(\mathbf{v} - \mathbf{v}) \leq \mathrm{abs}(\mathbf{v} + \mathbf{v}) \leq 2 * \mathrm{abs}(\mathbf{0})$$

For any non−zero value for abs(**0**) the inequalities would not hold. Only abs(**0**) = 0 works.

Newton: "So you have proven abs(**0**) = 0!

Breton: "Now let me ask you another question. Since every vector can be written

$$\mathbf{v} = \mathrm{abs}(\mathbf{v}) * \mathbf{u}(\mathbf{v})$$

that is, the scalar product of a magnitude and a direction, what is the direction of **0**?

Einstein: "The zero vector like the zero in the numbers is very special.

Breton: "Is it? Notice

$$\mathbf{0} = \mathrm{abs}(\mathbf{0}) * \mathbf{u}(\mathbf{0})$$
$$= 0 * \mathbf{u}(\mathbf{0})$$

so it appears any direction will do for **u**(**0**).

Einstein: "Breton, you never change your rascally ways.

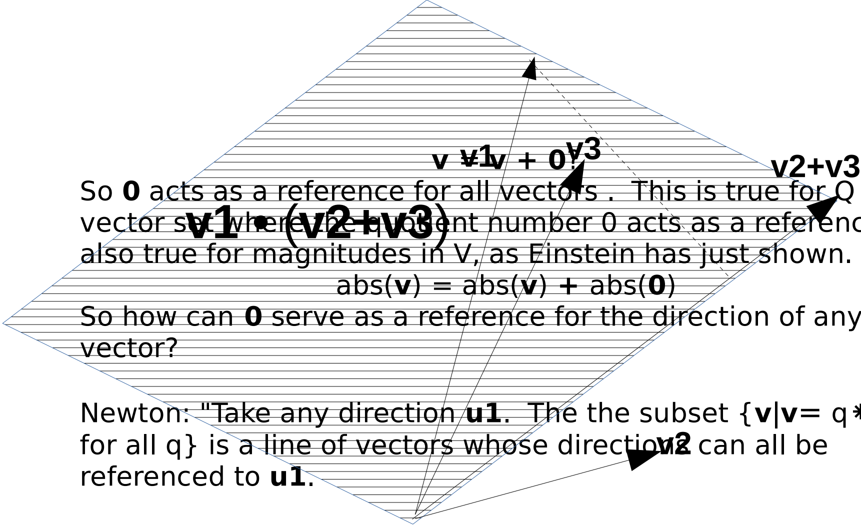
Breton: "Well if **u**(**0**) is any one of many directions that **0** can be any one of many vectors.

Newton: "But still it *is* only one vector.

Einstein: "What a morass you have led us into, Breton. It has to be one of many possible unspecified vectors.

Newton: "In fact any one of an infinite number of vectors, since directions correspond to all the points on the unit sphere.

Breton: "Remember the axiom concerning **0**



v = **v** + **0**!

So **0** acts as a reference for all vectors . This is true for Q as a vector set where the quotient number 0 acts as a reference; it also true for magnitudes in V, as Einstein has just shown.

abs(**v**) = abs(**v**) + abs(**0**)

So how can **0** serve as a reference for the direction of any vector?

Newton: "Take any direction **u1**. The the subset {**v**|**v**= q∗**u1** for all q} is a line of vectors whose directions can all be referenced to **u1**.

Breton: "Splendid. So **u1** can be part of our answer.

Newton: "Next take **u2** any direction orthogonal to **u1**. The the subset {**v**|**v**= q1∗**u1** + q2∗**u2** for all q1 and q2} is a plane of vectors whose directions can all be referenced to **u1** and **u2**.

Breton: "Splendid. So **u1** and **u2** can be part of our answer.

Newton: "Next take **u3** any direction orthogonal to **u1** and **u2**. The the subset {**v**|**v**= q1∗**u1** + q2∗**u2** + q3∗**u3** for all q1 and q2 and q3} is the set of all vectors each of which can be referenced to **u1** and **u2** and **u3**.

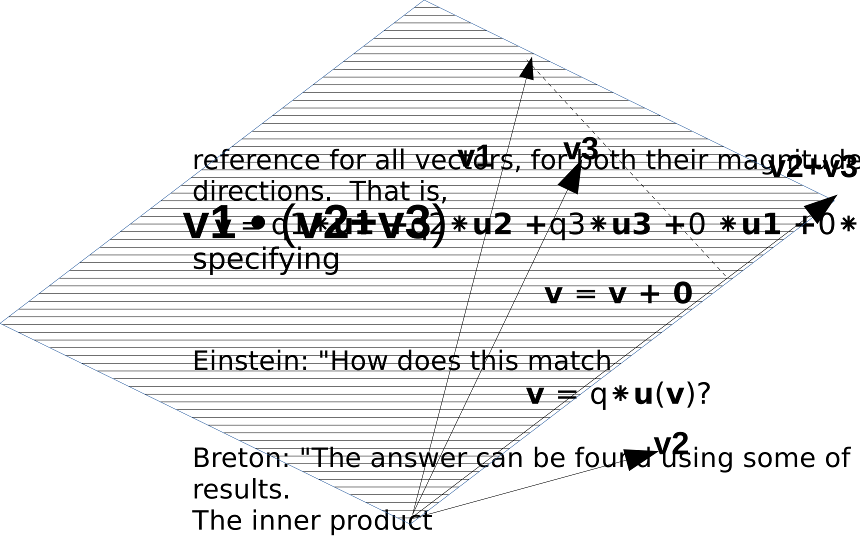
Einstein: "So in Newton's scheme we first choose arbitrarily one direction **u1** of all the directions of the sphere; next we choose arbitrarily a second direction **u2** from a great circle of the sphere orthogonal to **u1**; finally we no longer choose but accept the single direction **u3** which is orthogonal to both **u1** and **u2**.

Any one of the three directions taken singly serves as a reference to a subset of vectors in a line. Any two together serve as a reference to a subset of vectors in a plane. All three together serve as a reference to any vector.

Breton: "Including 0?

Einstein: "Yes. I like the balance between the choosing and the application.

Breton: "So now we have a reference for directions which though not derived from **0** will comport well with calling **0** the



reference for all vectors, for both their magnitudes and directions. That is,

v = q1∗**u1** +q2∗**u2** +q3∗**u3** +0 ∗**u1** +0∗**u2** +0∗**u3**

specifying

$$\mathbf{v} = \mathbf{v} + \mathbf{0}$$

Einstein: "How does this match

$$\mathbf{v} = q * \mathbf{u}(\mathbf{v})?$$

Breton: "The answer can be found using some of our previous results.
The inner product

$$\mathbf{v} \bullet \mathbf{v} = q^2$$

for **v** = q∗**u**(**v**)
and

$$\begin{aligned}
\mathbf{v} \bullet \mathbf{v} &= (q1*\mathbf{u1} + q2*\mathbf{u2} + q3*\mathbf{u3}) \\
&\quad \bullet (q1*\mathbf{u1} + q2*\mathbf{u2} + q3*\mathbf{u3}) \\
&= q1*\mathbf{u1}\bullet(q1*\mathbf{u1}) \\
&\quad + q1*\mathbf{u1}\bullet(q2*\mathbf{u2}) \\
&\quad + q1*\mathbf{u1}\bullet(q3*\mathbf{u3}) \\
&\quad + q2*\mathbf{u2}\bullet(q1*\mathbf{u1}) \\
&\quad + q2*\mathbf{u2}\bullet(q2*\mathbf{u2}) \\
&\quad + q2*\mathbf{u2}\bullet(q3*\mathbf{u3}) \\
&\quad + q31*\mathbf{u3}\bullet(q1*\mathbf{u1}) \\
&\quad + q3*\mathbf{u3}\bullet(q2*\mathbf{u2}) \\
&\quad + q3*\mathbf{u3}\bullet(q3*\mathbf{u3}) \\
&= q1*q1 + q2*q2 + q3*q3
\end{aligned}$$

since **ui•uj** = 0 if I≠j and **ui•uj** = 1 if i=j.

Newton: " So

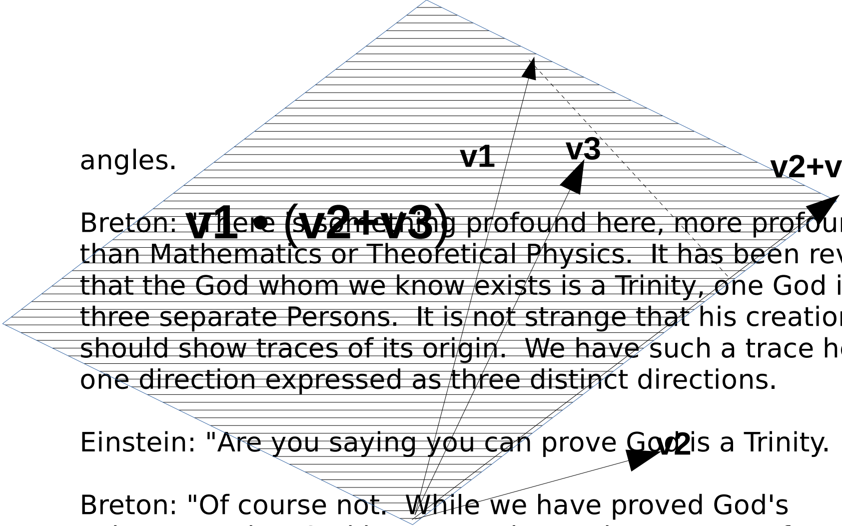
$$q^2 = q1^2 + q2^2 + q3^2$$

Breton: "By specifying **0** this way, we have made it a reference for all vectors, both their magnitudes and directions.

Einstein: "So this new **0** is different from the axiomatic **0**.

Breton: "To avoid ambiguity, let us call the new **0** the **origin** of our set of vectors.

Newton: "The origin fits in with our previous discussion of direction and angles. One direction can be specified by three



angles.

Breton: "There is something profound here, more profound than Mathematics or Theoretical Physics. It has been revealed that the God whom we know exists is a Trinity, one God in three separate Persons. It is not strange that his creation should show traces of its origin. We have such a trace here: one direction expressed as three distinct directions.

Einstein: "Are you saying you can prove God is a Trinity.

Breton: "Of course not. While we have proved God's existence, what God is appears beyond our power of comprehension. But if God reveals himself as Trinity, then the world becomes more comprehensible.

Einstein: "We defined Physics as a science which deals with a world that is extended, moving, and forcing. The Trinity is none of these things. So God, the Trinity, is not physical, and so not the object of scrutiny by the science of Physics.

Breton: "I agree. God is like a frame around a picture. God gives meaning to the science of Physics, but is not the direct object of its study. Like a frame around a picture.

Newton: "A most interesting subject I agree, but not on our path up the mountain. Is there more about the origin?

Breton: "Let's examine how a direction is expressed in terms of the origin's reference.

Einstein: "That's easy enough. Given the origin as described above, any direction

$$\mathbf{u}(\mathbf{v}) = q1 * \mathbf{u1} + q2 * \mathbf{u2} + q3 * \mathbf{u3})$$

for some quotient numbers, q1, q2, and q3 where

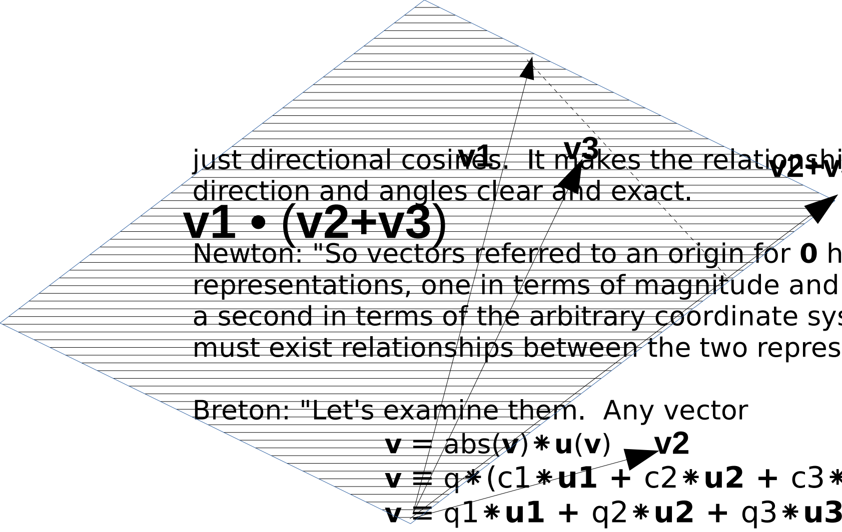
$$\mathrm{sqrt}(q1^2 + q2^2 + q3^2) = 1$$

from what we have just proven.

Breton: "For directions may I suggest replacing the symbol *q* with the symbol *c*.

Newton: "Why?

Breton: "Because from a geometrical perspective, the c's are



just directional cosines. It makes the relationship between direction and angles clear and exact.

Newton: "So vectors referred to an origin for **0** have two representations, one in terms of magnitude and direction, and a second in terms of the arbitrary coordinate system. There must exist relationships between the two representations.

Breton: "Let's examine them. Any vector

$$\mathbf{v} = \mathrm{abs}(\mathbf{v}) * \mathbf{u}(\mathbf{v})$$
$$\mathbf{v} \equiv q * (c1 * \mathbf{u1} + c2 * \mathbf{u2} + c3 * \mathbf{u3})$$
$$\mathbf{v} \equiv q1 * \mathbf{u1} + q2 * \mathbf{u2} + q3 * \mathbf{u3}$$
$$\mathbf{v} = \mathbf{v} \bullet \mathbf{u1} * \mathbf{u1} + \mathbf{v} \bullet \mathbf{u2} * \mathbf{u2} + \mathbf{v} \bullet \mathbf{u3} * \mathbf{u3}$$
$$\mathbf{v} = \mathbf{v} \bullet (\mathbf{u1} * \mathbf{u1} + \mathbf{u2} * \mathbf{u2} + \mathbf{u3} * \mathbf{u3})$$

The first of these relationships we have from the axioms;
the second is a definition of q as abs(**v**) for the magnitude of **v** with reference to the origin;
the third defines three magnitudes in the origin's directions;
the fourth equation relates these three magnitudes to the inner products with the given vector;
the fifth equation factors the fourth equation and establishes

$$\mathbf{I} \equiv \mathbf{u1} * \mathbf{u1} + \mathbf{u2} * \mathbf{u2} + \mathbf{u3} * \mathbf{u3}$$

as the identity transformation.
It follows that

$$qi = \mathbf{v} \bullet \mathbf{u1}$$
$$q = \mathrm{sqrt}(q1^2 + q2^2 + q3^2)$$
$$ci = qi/q$$

The three ci's are called directional cosines.

Einstein: "For a direction then

$$\mathrm{abs}(\mathbf{u}(\mathbf{v})) = \mathrm{sqrt}(q1^2 + q2^2 + q3^2) = 1$$

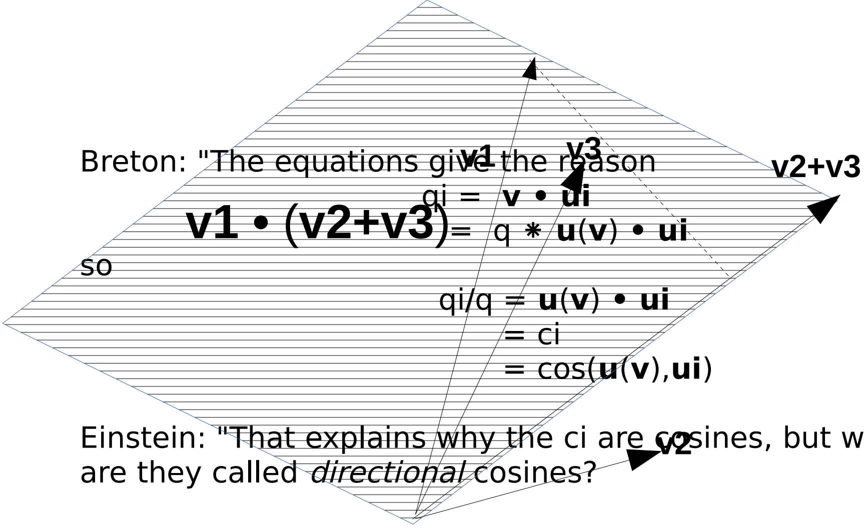
Newton: "Just as we asserted earlier. For any vector

$$\mathbf{v} = \mathrm{abs}(\mathbf{v}) * \mathbf{u}(\mathbf{v})$$
$$= q1 * \mathbf{u1} + q2 * \mathbf{u2} + q3 * \mathbf{u3}$$
$$= q * (c1 * \mathbf{u1} + c2 * \mathbf{u2} + c3 * \mathbf{u3})$$

so

$$q1 = q * c1$$
$$q2 = q * c2$$
$$q3 = q * c3$$

So tell us why the ci are called direction cosines.



Breton: "The equations give the reason

$$\begin{aligned} q_i &= \mathbf{v} \bullet \mathbf{u_i} \\ &= q * \mathbf{u}(\mathbf{v}) \bullet \mathbf{u_i} \end{aligned}$$

so

$$\begin{aligned} q_i/q &= \mathbf{u}(\mathbf{v}) \bullet \mathbf{u_i} \\ &= c_i \\ &= \cos(\mathbf{u}(\mathbf{v}),\mathbf{u_i}) \end{aligned}$$

Einstein: "That explains why the ci are cosines, but why are they called *directional* cosines?

Breton: "The angles of the three cosines define the vector's direction.

Newton: "How does representation in terms of the origin match with the vectorial operations.

Breton: "Easily enough. For

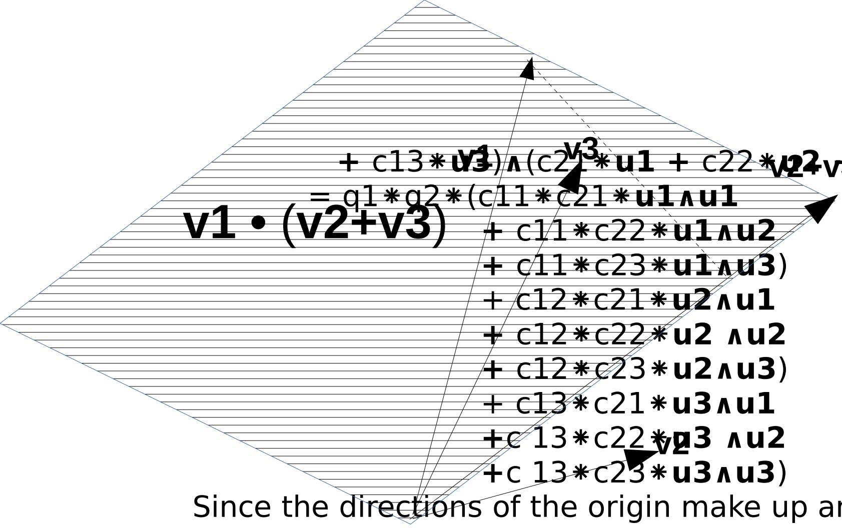
$$\begin{aligned} \mathbf{v1} &= q1*\mathbf{uv1} \\ &= q1*(c11*\mathbf{u1} + c12*\mathbf{u2} + c13*\mathbf{u3}) \\ \mathbf{v2} &= q2*\mathbf{uv2} \\ &= q2*(c21*\mathbf{u1} + c22*\mathbf{u2} + c23*\mathbf{u3}) \end{aligned}$$

$$\begin{aligned} \mathbf{v1+v2} &= (q1*c11+ q2*c21)*\mathbf{u1} \\ &\quad + (q1*c12+ q2*c22)*\mathbf{u2} \\ &\quad + (q1*c13+ q2*c23)*\mathbf{u3} \\ \mathbf{v1 \bullet v2} &= q1*q2*(c11*c21 + c12*c22 + c13*c23) \\ \mathbf{v1 \wedge v2} &= q1*q2*((c12*c23 - c13*c22)*\mathbf{u1} \\ &\quad + (c13*c21 - c11*c23)*\mathbf{u2} \\ &\quad + (c11*c22 - c12*c21)*\mathbf{u3}) \\ \mathbf{v1 * v2} &= q1*q2*\mathbf{uv1*uv2} \end{aligned}$$

Einstein: "Wherever did you get $\mathbf{v1 \wedge v2}$?

Breton: "I will now answer your question about the ambiguity in the cross product. Please follow these straight–forward substitutions and operations.

$$\begin{aligned} \mathbf{v1 \wedge v2} &= q1*q2*\mathbf{uv1 \wedge uv2} \\ &= q1*q2* \\ &\quad (c11*\mathbf{u1} \wedge (c21*\mathbf{u1} + c22*\mathbf{u2} + c23*\mathbf{u3}) \\ &\quad + c12*\mathbf{u2} \wedge (c21*\mathbf{u1} + c22*\mathbf{u2} + c23*\mathbf{u3}) \end{aligned}$$



+ c13∗**u3**)∧(c21∗**u1** **+** c22∗**u2** + c23∗**u3**)
= q1∗q2∗(c11∗c21∗**u1∧u1**
+ c11∗c22∗**u1∧u2**
+ c11∗c23∗**u1∧u3**)
+ c12∗c21∗**u2∧u1**
+ c12∗c22∗**u2 ∧u2**
+ c12∗c23∗**u2∧u3**)
+ c13∗c21∗**u3∧u1**
+c 13∗c22∗**u3 ∧u2**
+c 13∗c23∗**u3∧u3**)

Since the directions of the origin make up an orthogonal set, the following definitions resolve the ambiguity in the vector product.

u1∧u2 ≡ **u3**
u2∧u3 ≡ **u1**
u3∧u1 ≡ **u2**

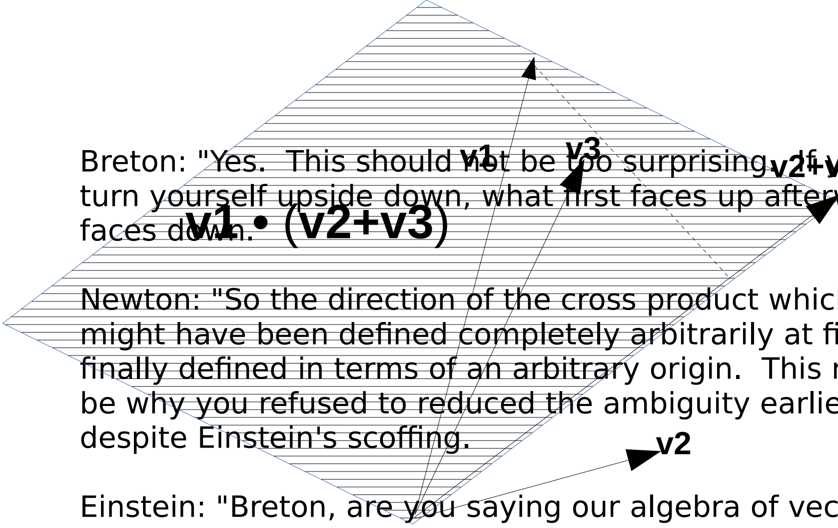
You can see that the cyclic arrangement I alluded to before is incorporated in these definitions.

Remembering that **ui∧ui** = **0** and that **ui∧uj** = **–uj∧ui**

v1∧v2 = q1∗q2∗(c11∗c22∗**u3**
– c11∗c23∗**u2**
– c12∗c21∗**u3**
+ c12∗c23∗**u1**
+ c13∗c21∗**u2**
– c13∗c22∗**u1**)
= q1∗q2∗(c12∗c23∗**u1**
– c13∗c22∗**u1**)
+ c13∗c21∗**u2**
– c11∗c23∗**u2**
+ c11∗c22∗**u3**
– c12∗c21∗**u3**
= q1∗q2∗((c12∗c23 – c13∗c22)∗**u1**)
+ (c13∗c21 – c11∗c23)∗**u2**
+ (c11∗c22 – c12∗c21)∗**u3**)

Newton: "Just as you stated.

Einstein: "So the direction of the cross product depends on the choice of origin.



Breton: "Yes. This should not be too surprising. If you turn yourself upside down, what first faces up afterwards faces down.

Newton: "So the direction of the cross product which might have been defined completely arbitrarily at first, is finally defined in terms of an arbitrary origin. This must be why you refused to reduced the ambiguity earlier, despite Einstein's scoffing.

Einstein: "Breton, are you saying our algebra of vector sets then applies only to a particular choice of origin.

Breton: "That is a question which separates your distinguished ancestors. Isaac Newton held that one special location in the universe is an absolute location. His discoveries depended on such an assumption. Albert Einstein disagreed. The origin of our algebra for vectors sets may illumine the controversy. So let us put the question on our agenda, but first let us clear this path about vectorial operations referred to the origin a little more.

Both agree. For different reasons.

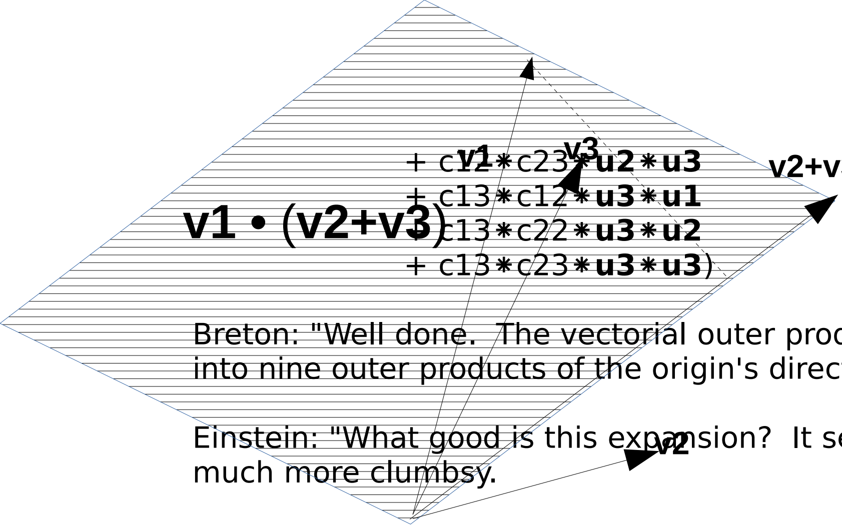
Breton: "We haven't expressed $\mathbf{v1}*\mathbf{v2}$ in terms of the origin.

Newton: "We said before

$$\mathbf{v1}*\mathbf{v2} = q1*q2*\mathbf{uv1}*\mathbf{uv2}$$

which is the vectorial expression. Referred to the origin this becomes

$$\begin{aligned}\mathbf{v1}*\mathbf{v2} &= q1*q2*(c11*\mathbf{u1} + c12*\mathbf{u2} + c13*\mathbf{u3})\\ &\qquad *(c12*\mathbf{u1} + c22*\mathbf{u2} + c23*\mathbf{u3})\\ &= q1*q2\\ &\qquad *(c11*c12*\mathbf{u1}*\mathbf{u1}\\ &\qquad + c11*c22*\mathbf{u1}*\mathbf{u2}\\ &\qquad + c11*c23*\mathbf{u1}*\mathbf{u3}\\ &\qquad + c12*c12*\mathbf{u2}*\mathbf{u1}\\ &\qquad + c12*c22*\mathbf{u2}*\mathbf{u2}\end{aligned}$$



+ c12∗c23∗**u2**∗**u3**
+ c13∗c12∗**u3**∗**u1**
+ c13∗c22∗**u3**∗**u2**
+ c13∗c23∗**u3**∗**u3**)

Breton: "Well done. The vectorial outer product explodes into nine outer products of the origin's directions.

Einstein: "What good is this expansion? It seems so much more clumbsy.

Breton: "It often makes proving propositions rather more simple. Let me illustrate We have shown for the triple scalar product.

(**v1 ∧ v2**)•**v3** =(**v2 ∧ v3**)•**v1**

basing the proof on geometry. Let's try a proof based on the origin. We would have

(**v1 ∧ v2**)•**v3**
= q1∗q2∗((c12∗c23 – c13∗c22)∗**u1**)
+ (c13∗c21 – c11∗c23)∗**u2**
+ (c11∗c22 – c12∗c21)∗**u3**)
•q3∗(c31∗**u1 +** c32∗**u2 +** c33∗**u3**)

where **v3** = q3∗(c31∗**u1 +** c32∗**u2 +** c33∗**u3**)

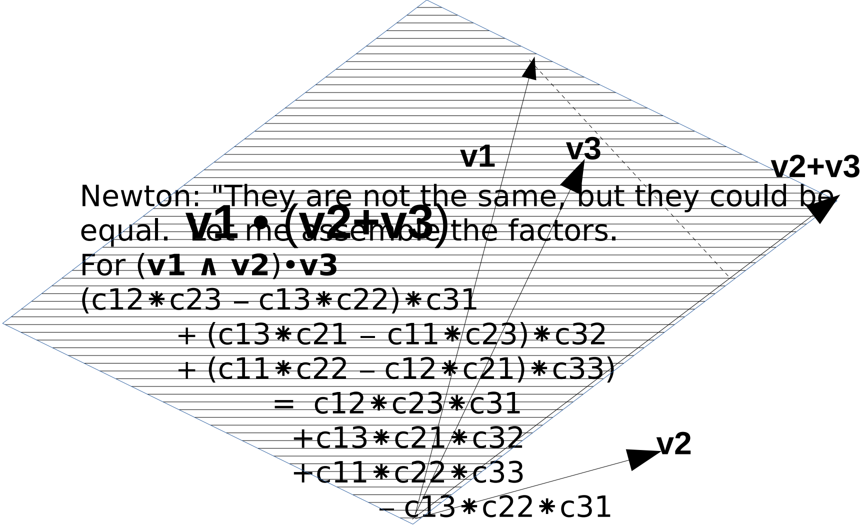
Then

(**v1 ∧ v2**)•**v3**
= q1∗q2∗q3
∗((c12∗c23 – c13∗c22)∗c31
+ (c13∗c21 – c11∗c23)∗c32
+ (c11∗c22 – c12∗c21)∗c33)

and

(**v2 ∧ v3**)•**v1**
= q2∗q3∗((c22∗c33 – c23∗c32)∗**u1**)
+ (c23∗c31 – c21∗c33)∗**u2**
+ (c21∗c32 – c22∗c31)∗**u3**)
•q1∗(c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
= q2∗q3∗q1
∗(c22∗c33 – c23∗c32)∗c11
+ (c23∗c31 – c21∗c33)∗c12
+ (c21∗c32 – c22∗c31)∗c13)

Are they equal?



Newton: "They are not the same, but they could be equal. Let me assemble the factors.

For (**v1 ∧ v2**)•**v3**

(c12∗c23 – c13∗c22)∗c31
+ (c13∗c21 – c11∗c23)∗c32
+ (c11∗c22 – c12∗c21)∗c33)
= c12∗c23∗c31
+c13∗c21∗c32
+c11∗c22∗c33
– c13∗c22∗c31
– c11∗c23∗c32
– c12∗c21∗c33)

For (**v2 ∧ v3**)•**v1**

(c22∗c33 – c23∗c32)∗c11
+ (c23∗c31 – c21∗c33)∗c12
+ (c21∗c32 – c22∗c31)∗c13)
= c22∗c33∗c11
+ c23∗c31∗c12
+ c21∗c32∗c13
– c23∗c32∗c11
–c21∗c33∗c12
– c22∗c31∗c13

So although the summands are differently ordered, each summand in one case finds its match in the other case.
And in both cases

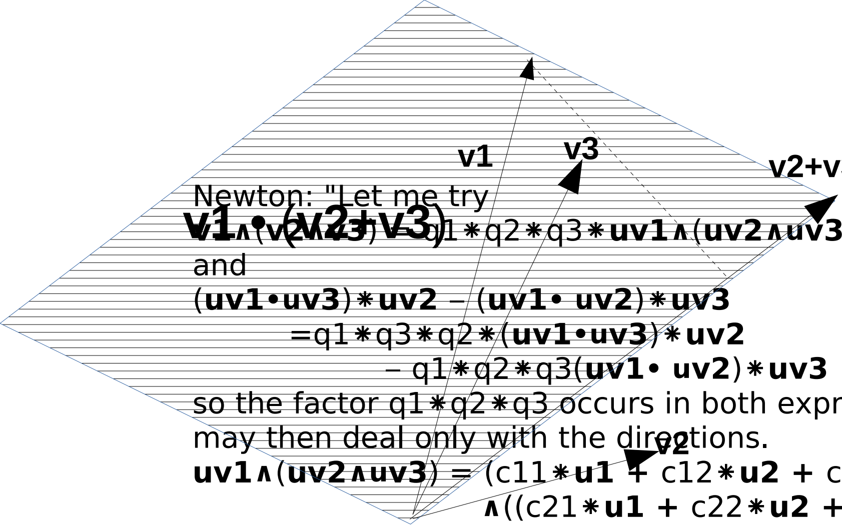
q1∗q2∗q3 = q2∗q3∗q1

So (**v2 ∧ v3**)•**v1** does indeed equal (**v1 ∧ v2**)•**v3**

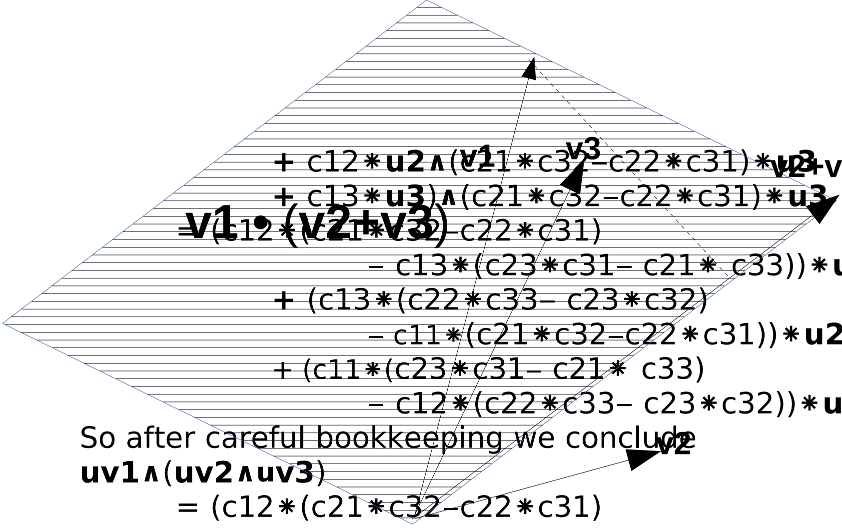
Einstein: "A proof without the difficult geometry.

Breton: "The origin with its orthogonal coordinates does not eliminate the geometry so much as finesses it. True geometrical propositions about areas and volumes difficult to prove by geometrical methods alone, may find easier proofs by vectorial algebra.

Einstein: "I am willing to concede that all the variations of the scalar triple product can be proven similarly, but what about the vector triple product?



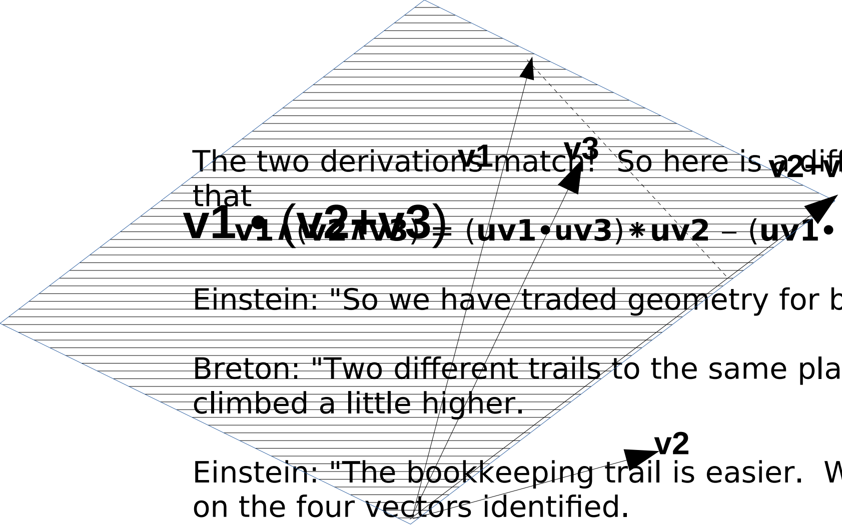
Newton: "Let me try
v1∧(**v2∧v3**) = q1∗q2∗q3∗**uv1∧**(**uv2∧uv3**)
and
(**uv1•uv3**)∗**uv2** – (**uv1• uv2**)∗**uv3**
=q1∗q3∗q2∗(**uv1•uv3**)∗**uv2**
– q1∗q2∗q3(**uv1• uv2**)∗**uv3**
so the factor q1∗q2∗q3 occurs in both expressions; we may then deal only with the directions.
uv1∧(**uv2∧uv3**) = (c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
∧((c21∗**u1 +** c22∗**u2 +** c23∗**u3**)
∧(c31∗**u1 +** c32∗**u2 +** c33∗**u3**))
= (c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
∧(c21∗**u1 ∧**c31∗**u1**
+ c22∗**u2∧**c31∗**u1**
+ c23∗**u3∧**c31∗**u1**
+ c21∗**u1 ∧**c32∗**u2**
+ c22∗**u2∧**c32∗**u2**
+ c23∗**u3∧**c32∗**u2**
+ c21∗**u1 ∧** c33∗**u3**
+ c22∗**u2∧**c c33∗**u3**
+ c23∗**u3∧** c33∗**u3**)
= (c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
∧(–c22c31∗**u3**
+ c23∗c31∗**u2**
+ c21∗c32∗**u3**
– c23∗c32∗**u1**
– c21∗ c33∗**u2**
+ c22∗c33∗**u1**)
= (c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
∧((c22∗c33– c23∗c32)∗**u1**
+ (c23∗c31– c21∗ c33)∗**u2**
+ (c21∗c32–c22∗c31)∗**u3**)
= (c11∗**u1∧**(c22∗c33– c23∗c32)∗**u1**
+ c12∗**u2∧**(c22∗c33– c23∗c32)∗**u1**
+ c13∗**u3**)**∧**(c22∗c33– c23∗c32)∗**u1**
+ c11∗**u1∧**(c23∗c31– c21∗ c33)∗**u2**
+ c12∗**u2∧**(c23∗c31– c21∗ c33)∗**u2**
+ c13∗**u3**)**∧**(c23∗c31– c21∗ c33)∗**u2**
+ c11∗**u1∧**(c21∗c32–c22∗c31)∗**u3**



\+ c12∗**u2∧**(c21∗c32–c22∗c31)∗**u3**
\+ c13∗**u3**)**∧**(c21∗c32–c22∗c31)∗**u3**
= (c12∗(c21∗c32–c22∗c31)
– c13∗(c23∗c31– c21∗ c33))∗**u1**
+ (c13∗(c22∗c33– c23∗c32)
– c11∗(c21∗c32–c22∗c31))∗**u2**
\+ (c11∗(c23∗c31– c21∗ c33)
– c12∗(c22∗c33– c23∗c32))∗**u3**

So after careful bookkeeping we conclude
uv1∧(**uv2∧uv3**)
= (c12∗(c21∗c32–c22∗c31)
– c13∗(c23∗c31– c21∗ c33))∗**u1**
+ (c13∗(c22∗c33– c23∗c32)
– c11∗(c21∗c32–c22∗c31))∗**u2**
\+ (c11∗(c23∗c31– c21∗ c33)
– c12∗(c22∗c33– c23∗c32))

Now let me calculate
(**uv1•uv3**)∗**uv2** – (**uv1• uv2**)∗**uv3**
=(c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
•(c31∗**u1 +** c32∗**u2 +** c33∗**u3**)
∗(c21∗**u1 +** c22∗**u2 +** c23)∗**u3**)
–(c11∗**u1 +** c12∗**u2 +** c13∗**u3**)
•(c21∗**u1 +** c22∗**u2 +** c23∗**u2**)
∗(c31∗**u1 +** c32∗**u2 +** c33)∗**u3**)
=(c11∗c31 **+** c12∗c32**+** c13∗c33)
∗(c21∗**u1 +** c22∗**u2 +** c23)∗**u3**)
–(c11∗ c21 **+** c12∗ c22 **+** c13∗c23) c21∗**u1**
∗(c31∗**u1 +** c32∗**u2 +** c33)∗**u3**)
=((c11∗c31 **+** c12∗c32**+** c13∗c33)∗c21
–(c11∗ c21 **+** c12∗ c22 **+** c13∗c23)∗c31)∗**u1**
+((c11∗c31 **+** c12∗c32**+** c13∗c33)∗c22
–(c11∗ c21 **+** c12∗ c22 **+** c13∗c23)∗c32)∗**u2**
+((c11∗c31 **+** c12∗c32**+** c13∗c33)∗c23
–(c11∗ c21 **+** c12∗ c22 **+** c13∗c23)∗c33)∗**u3**
=(c12∗c32∗c21 – c12∗ c22∗c31
+ c13∗c33∗c21 –c13∗c23∗c31)∗**u1**
+((c11∗c31∗c22 –c11∗ c21∗c32
+ c13∗c33∗c22 – c13∗c23∗c32)∗**u2**
+(c11∗c31∗c23 –c11∗ c21∗c33
+c12∗c32∗c23–c12∗ c22∗c33)∗**u3**



The two derivations match! So here is a different proof that

$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) = (\mathbf{uv1}\bullet\mathbf{uv3})*\mathbf{uv2} - (\mathbf{uv1}\bullet\mathbf{uv2})*\mathbf{uv3}$

Einstein: "So we have traded geometry for bookkeeping.

Breton: "Two different trails to the same place. We have climbed a little higher.

Einstein: "The bookkeeping trail is easier. Why not try it on the four vectors identified.

Newton: "Let's pick up where we were before. Does
$(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4}) = \mathbf{v1}\bullet(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$?
Let
$\mathbf{v1}=q1*(c11*\mathbf{u1} + c12*\mathbf{u2} + c13*\mathbf{u3})$
$\mathbf{v2}=q2*(c21*\mathbf{u1} + c22*\mathbf{u2} + c23*\mathbf{u3})$
$\mathbf{v3}=q3*(c31*\mathbf{u1} + c32*\mathbf{u2} + c33*\mathbf{u3})$
$\mathbf{v4}=q4*(c41*\mathbf{u1} + c42*\mathbf{u2} + c43*\mathbf{u3})$
Then

$$\begin{aligned}\mathbf{v1}\wedge\mathbf{v2} = q1*q2*(&(c12*c23 - c13*c22)*\mathbf{u1})\\ &+ (c13*c21 - c11*c23)*\mathbf{u2}\\ &+ (c11*c22 - c12*c21)*\mathbf{u3})\end{aligned}$$

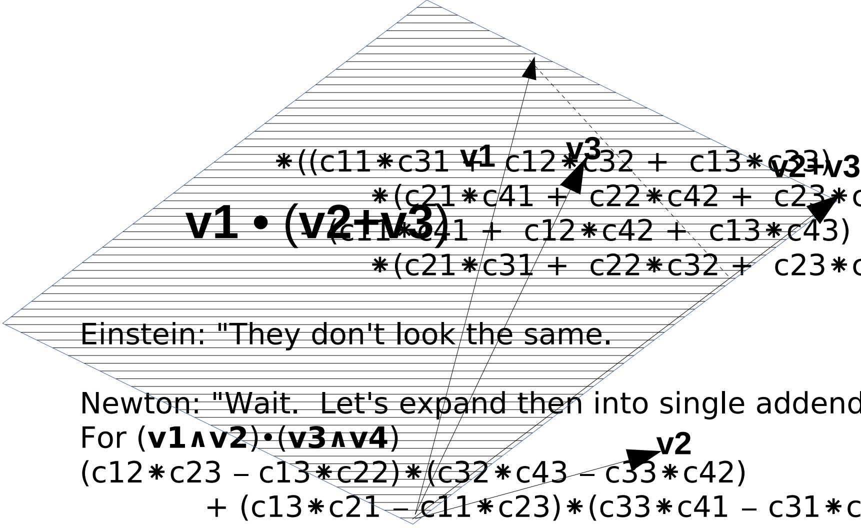
$$\begin{aligned}\mathbf{v3}\wedge\mathbf{v4} = q3*q4*(&(c32*c43 - c33*c42)*\mathbf{u1})\\ &+ (c33*c41 - c31*c43)*\mathbf{u2}\\ &+ (c31*c42 - c32*c41)*\mathbf{u3})\end{aligned}$$

so

$$\begin{aligned}(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4}) = q1*q2*q3*q4\\ &*((c12*c23 - c13*c22)*\mathbf{u1})\\ &\quad + (c13*c21 - c11*c23)*\mathbf{u2}\\ &\quad + (c11*c22 - c12*c21)*\mathbf{u3})\\ &\bullet ((c32*c43 - c33*c42)*\mathbf{u1})\\ &\quad + (c33*c41 - c31*c43)*\mathbf{u2}\\ &\quad + (c31*c42 - c32*c41)*\mathbf{u3})\\ =(c12*c23 - c13*c22)&*(c32*c43 - c33*c42)\\ + (c13*c21 - c11*c23)&*(c33*c41 - c31*c43)\\ + (c11*c22 - c12*c21)&* (c31*c42 - c32*c41)\end{aligned}$$

while

$$\begin{aligned}(\mathbf{v1}\bullet\mathbf{v3})*(\mathbf{v2}\bullet\mathbf{v4}) - (\mathbf{v1}\bullet\mathbf{v4})*(\mathbf{v2}\bullet\mathbf{v3})&\\ = q1*q2*q3*q4&\end{aligned}$$

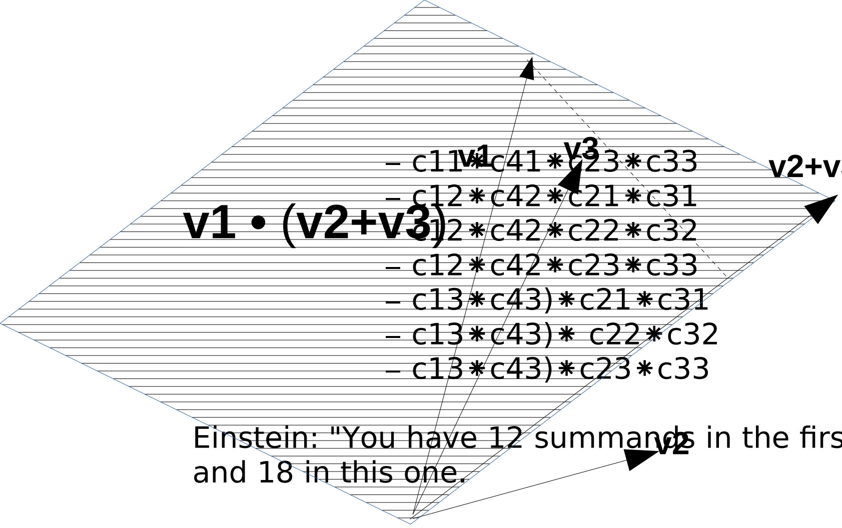


∗((c11∗c31 + c12∗c32 + c13∗c33)
∗(c21∗c41 + c22∗c42 + c23∗c43)
(c11∗c41 + c12∗c42 + c13∗c43)
∗(c21∗c31 + c22∗c32 + c23∗c33))

Einstein: "They don't look the same.

Newton: "Wait. Let's expand then into single addends.
For (**v1∧v2**)•(**v3∧v4**)
(c12∗c23 – c13∗c22)∗(c32∗c43 – c33∗c42)
+ (c13∗c21 – c11∗c23)∗(c33∗c41 – c31∗c43)
+ (c11∗c22 – c12∗c21)∗(c31∗c42 – c32∗c41)
=(c12∗c23∗c32∗c43
+ c13∗c22∗c33∗c42
–c12∗c23∗c33∗c42
– c13∗c22∗c32∗c43
+ c13∗c21∗c33∗c41
+ c11∗c23∗c31∗c43
–c13∗c21∗c31∗c43
– c11∗c23∗c33∗c41
+ c11∗c22∗c31∗c42
+ c12∗c21∗c32∗c41
–c11∗c22∗c32∗c41
– c12∗c21∗c31∗c42
while for (**v1•v3**)∗(**v2•v4**) – (**v1•v4**)∗(**v2•v3**)
((c11∗c31 + c12∗c32 + c13∗c33)
∗(c21∗c41 + c22∗c42 + c23∗c43)
– (c11∗c41 + c12∗c42 + c13∗c43)
∗(c21∗c31 + c22∗c32 + c23∗c33))
= c11∗c31∗c21∗c41
+ c11∗c31∗c22∗c42
+ c11∗c31∗c23∗c43
+ c12∗c32∗c21∗c41
+ c12∗c32∗ c22∗c42
+ c12∗c32∗c23∗c43
+ c13∗c33∗c21∗c41
+ c13∗c33∗c22∗c42
+ c13∗c33∗c23∗c43
– c11∗c41∗c21∗c31
– c11∗c41∗c22∗c32

– c11∗c41∗c23∗c33
– c12∗c42∗c21∗c31
– c12∗c42∗c22∗c32
– c12∗c42∗c23∗c33
– c13∗c43)∗c21∗c31
– c13∗c43)∗ c22∗c32
– c13∗c43)∗c23∗c33



Einstein: "You have 12 summands in the first compilation and 18 in this one.

Newton: "But six of them cancel; so this final compilation also reduces to 12 summands which are
= c11∗c31∗c22∗c42
+ c11∗c31∗c23∗c43
+ c12∗c32∗c21∗c41
+ c12∗c32∗c23∗c43
+ c13∗c33∗c21∗c41
+ c13∗c33∗c22∗c42
– c11∗c41∗c22∗c32
– c11∗c41∗c23∗c33
– c12∗c42∗c21∗c31
– c12∗c42∗c23∗c33
– c13∗c43∗c21∗c31
– c13∗c43∗ c22∗c32
Now check these.

Einstein: "They all match! I'm amazed. We've traded geometry for bookkeeping, and the bookkeeping seems easier.

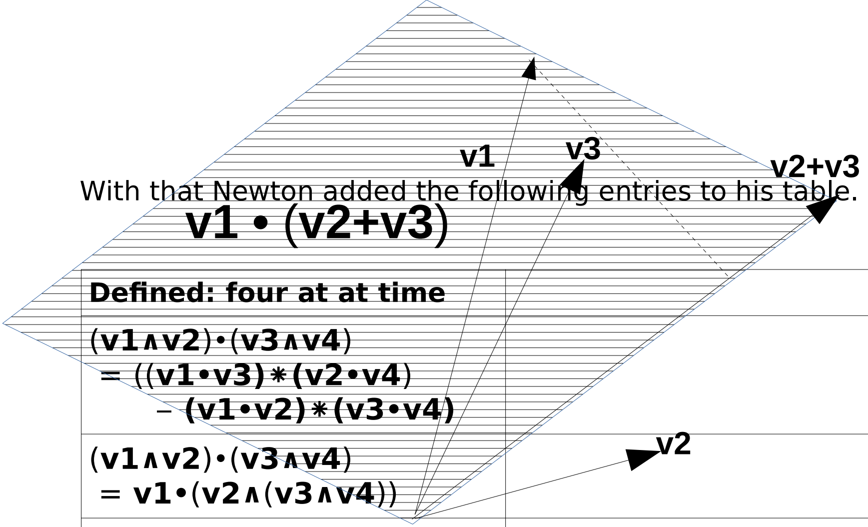
Breton: "So Newton now you have a couple more entries for your table.

Newton: "Only one.

Breton: " Acutally two. Remember we proved earlier
$\mathbf{v1}\bullet(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4})) = (\mathbf{v1}\bullet\mathbf{v3})\ast(\mathbf{v2}\bullet\mathbf{v4}) - (\mathbf{v1}\bullet\mathbf{v4})\ast(\mathbf{v2}\bullet\mathbf{v3})$
so we also know

$$(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4}) = \mathbf{v1}\bullet(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$$

With that Newton added the following entries to his table.



v1 • (v2+v3)

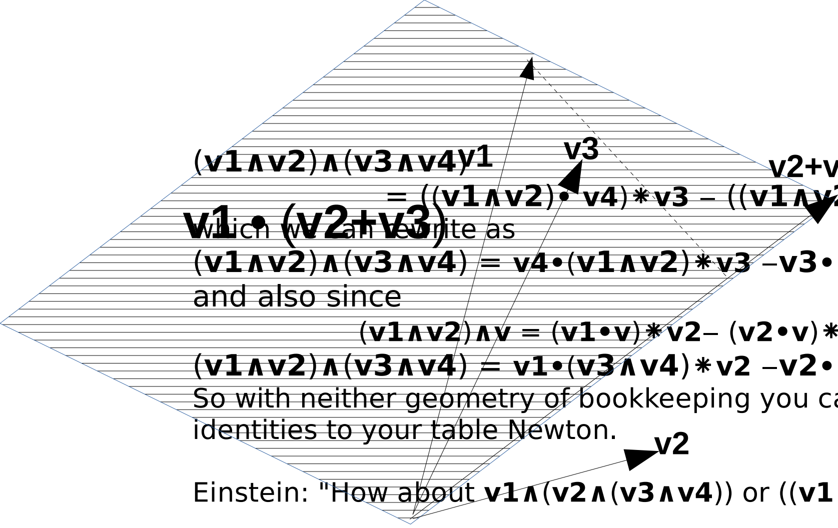
Defined: four at at time	
(**v1∧v2**)•(**v3∧v4**) = ((**v1•v3)⁎(v2•v4**) – **(v1•v2)⁎(v3•v4)**	
(**v1∧v2**)•(**v3∧v4**) = **v1**•(**v2∧**(**v3∧v4**))	
(**v1∧v2**)•(**v3∧v4**) + (**v1∧v3**)•(**v4∧v2**) + (**v1∧v4**)•(**v2∧v3**) = 0	

Einstein: "You've added still more entries.

Newton: "And I could have easily added still others. For instance
since (**v1∧v2**)•(**v3∧v4**)
= ((**v1•v3)⁎(v2•v4**) – **(v1•v2)⁎(v3•v4)**
then
(**v1∧v3**)•(**v4∧v2**)
= ((**v1•v4)⁎(v3•v2**) – **(v1•v3)⁎(v4•v2)**
and
(**v1∧v4**)•(**v2∧v3**)
= ((**v1•v2)⁎(v4•v3**) – **(v1•v4)⁎(v2•v3)**
each of which is proved by as mere substitution of labels.
Sum them together; you will find each positive summand matched by a negative summand.

Einstein: "Other possibilities exist. How about (**v1∧v2**)**∧**(**v3∧v4**)?

Breton: "That's already solved. Remember
v∧(**v2∧v3**) = (**v•v3**)⁎**v2** – (**v• v2**)⁎**v3**
from above? By simply relabelling
v2 as **v3**
v3 as **v4**
v as **v1∧v2**)
the equation becomes



$$(\mathbf{v1}\wedge\mathbf{v2})\wedge(\mathbf{v3}\wedge\mathbf{v4})$$
$$= ((\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v4})\ast\mathbf{v3} - ((\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v3})\ast\mathbf{v2}$$

which we can rewrite as

$$(\mathbf{v1}\wedge\mathbf{v2})\wedge(\mathbf{v3}\wedge\mathbf{v4}) = \mathbf{v4}\bullet(\mathbf{v1}\wedge\mathbf{v2})\ast\mathbf{v3} - \mathbf{v3}\bullet(\mathbf{v1}\wedge\mathbf{v2})\ast\mathbf{v2}$$

and also since

$$(\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v} = (\mathbf{v1}\bullet\mathbf{v})\ast\mathbf{v2} - (\mathbf{v2}\bullet\mathbf{v})\ast\mathbf{v1}$$

$$(\mathbf{v1}\wedge\mathbf{v2})\wedge(\mathbf{v3}\wedge\mathbf{v4}) = \mathbf{v1}\bullet(\mathbf{v3}\wedge\mathbf{v4})\ast\mathbf{v2} - \mathbf{v2}\bullet(\mathbf{v3}\wedge\mathbf{v4})\ast\mathbf{v1}$$

So with neither geometry of bookkeeping you can add these identities to your table Newton.

Einstein: "How about $\mathbf{v1}\wedge(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$ or $((\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3})\wedge\mathbf{v4}$?

Newton: "Let me try these. We know

$$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v}) = (\mathbf{v1}\bullet\mathbf{v})\ast\mathbf{v2} - (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v}$$

so

$$\mathbf{v1}\wedge(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$$
$$= (\mathbf{v1}\bullet(\mathbf{v3}\wedge\mathbf{v4}))\ast\mathbf{v2} - (\mathbf{v1}\bullet\mathbf{v2})\ast(\mathbf{v3}\wedge\mathbf{v4})$$

and likewise since we know

$$(\mathbf{v}\wedge\mathbf{v3})\wedge\mathbf{v4} = (\mathbf{v}\bullet\mathbf{v4})\ast\mathbf{v3} - (\mathbf{v3}\bullet\mathbf{v4})\ast\mathbf{v}$$

$$((\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3})\wedge\mathbf{v4}$$
$$= ((\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v4}))\ast\mathbf{v3} - (\mathbf{v3}\bullet\mathbf{v4})\ast(\mathbf{v1}\wedge\mathbf{v2})$$

Easy. Indeed, I begin to see relationships between our results. Look

$$\mathbf{v1}\wedge(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$$
$$= (\mathbf{v1}\bullet(\mathbf{v3}\wedge\mathbf{v4}))\ast\mathbf{v2} - (\mathbf{v1}\bullet\mathbf{v2})\ast(\mathbf{v3}\wedge\mathbf{v4})$$

so

$$\mathbf{v1}\wedge((\mathbf{v3}\wedge\mathbf{v4})\wedge\mathbf{v2})$$
$$= (\mathbf{v1}\bullet\mathbf{v2})\ast(\mathbf{v3}\wedge\mathbf{v4}) - (\mathbf{v1}\bullet(\mathbf{v3}\wedge\mathbf{v4}))\ast\mathbf{v2}$$

which can be rewritten

$$\mathbf{v1}\wedge((\mathbf{v2}\wedge\mathbf{v3})\wedge\mathbf{v4})$$
$$= (\mathbf{v1}\bullet\mathbf{v3})\ast(\mathbf{v4}\wedge\mathbf{v2}) - (\mathbf{v1}\bullet(\mathbf{v4}\wedge\mathbf{v2}))\ast\mathbf{v3}$$
$$= (\mathbf{v1}\bullet(\mathbf{v2}\wedge\mathbf{v4}))\ast\mathbf{v3} - (\mathbf{v1}\bullet\mathbf{v3})\ast(\mathbf{v2}\wedge\mathbf{v4})$$

and likewise

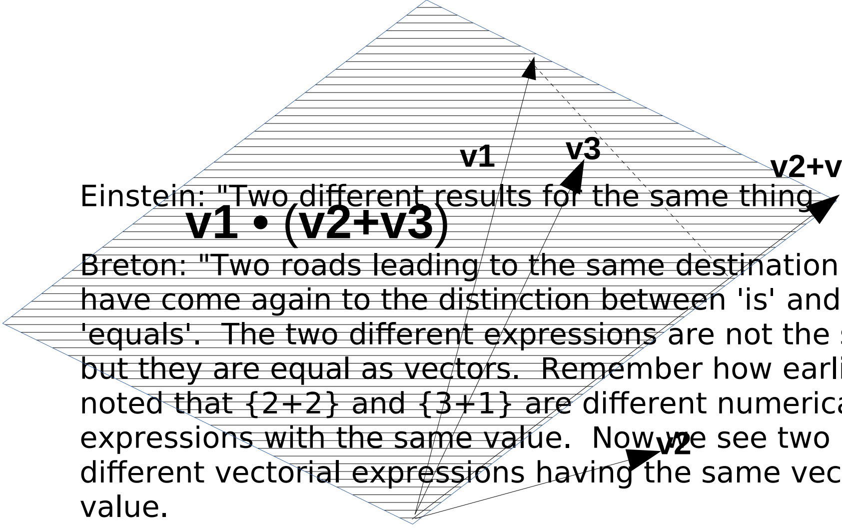
$$((\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3})\wedge\mathbf{v4}$$
$$= ((\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v4}))\ast\mathbf{v3} - (\mathbf{v3}\bullet\mathbf{v4})\ast(\mathbf{v1}\wedge\mathbf{v2})$$

so

$$(\mathbf{v3}\wedge(\mathbf{v1}\wedge\mathbf{v2}))\wedge\mathbf{v4}$$
$$= (\mathbf{v3}\bullet\mathbf{v4})\ast(\mathbf{v1}\wedge\mathbf{v2}) - ((\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v4}))\ast\mathbf{v3}$$

which can be rewritten

$$(\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}))\wedge\mathbf{v4}$$
$$= (\mathbf{v1}\bullet\mathbf{v4})\ast(\mathbf{v2}\wedge\mathbf{v3}) - ((\mathbf{v2}\wedge\mathbf{v3})\bullet\mathbf{v4}))\ast\mathbf{v1}$$



Einstein: "Two different results for the same thing.

Breton: "Two roads leading to the same destination. We have come again to the distinction between 'is' and 'equals'. The two different expressions are not the same, but they are equal as vectors. Remember how earlier we noted that {2+2} and {3+1} are different numerical expressions with the same value. Now we see two different vectorial expressions having the same vectorial value.

Newton: "Another breathtaking intellectual vista.

Breton: "Just as with numbers, the many expressions for same value lead to equations and eventually to an algebra. We can note here that from the above we have
(**v1•(v2∧v4)**)∗**v3** – (**v1• v3**)∗**(v2∧v4)**
= (**v1• v4**)∗**(v2∧v3)** – ((**v2∧v3**)**•v4)**)∗**v1**
which becomes on transposing,
(**v1•(v2∧v4)**)∗**v3 + (**(**v2∧v3**)**•v4))**∗**v1**
= (**v1• v4**)∗**(v2∧v3)** + (**v1• v3**)∗**(v2∧v4)**

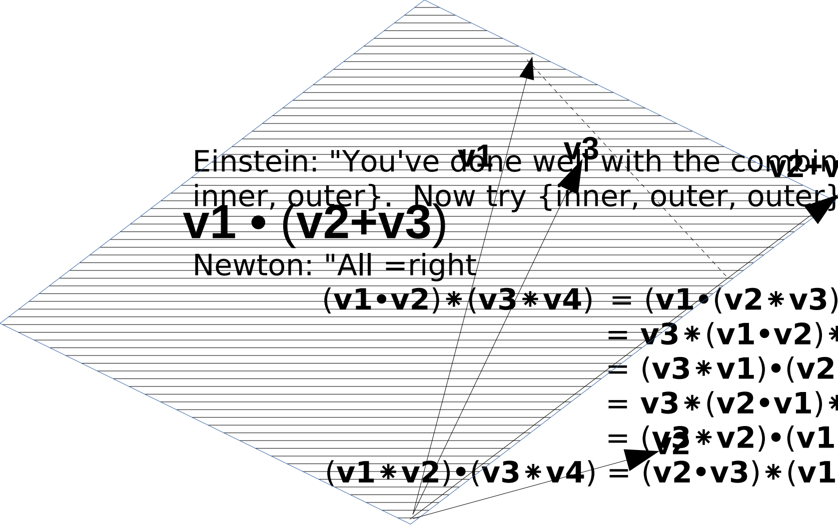
Einstein: "Let me observe that with four vectors we can insert three different multiplications.

Breton: "But not all of these are legitimate. For instance, **v1•v2•v3•v4** makes no sense.

Einstein: "While **v1•**(**v2•v3**∗**v4**) does. So let us continue with other possibilities.

Newton: "Some are easy enough. Let me write some for you.

v1•(**v2•v3**∗**v4**) = (**v1•v4**)∗(**v2•v3**)
v1•(**v2**∗**v3**)**•v4** = (**v1•v2**)∗(**v3•v4**)
= **v1•**(**v2**∗**v4**)**•v3**
= **v2•**(**v1**∗**v3**)**•v4**
= **v2•**(**v1**∗**v4**)**•v3**



Einstein: "You've done well with the combination {inner, inner, outer}. Now try {inner, outer, outer}!

Newton: "All =right

(**v1**•**v2**)∗(**v3**∗**v4**) = (**v1**•(**v2**∗**v3**))∗**v4**
= **v3**∗(**v1**•**v2**)∗**v4**
= (**v3**∗**v1**)•(**v2**∗**v4**)
= **v3**∗(**v2**•**v1**)∗**v4**
= (**v3**∗**v2**)•(**v1**∗**v4**)
(**v1**∗**v2**)•(**v3**∗**v4**) = (**v2**•**v3**)∗(**v1**∗**v4**)

Einstein: "The last equation is not obvious.

Newton: "It's simple enough. For any vector **v**

v•(**v1**∗**v2**)•(**v3**∗**v4**) = (**v**•**v1**)∗**v2**•(**v3**∗**v4**)
= (**v**•**v1**)∗(**v2**•**v3**)∗**v4**)
=(**v2**•**v3**)∗(**v**•**v1**)∗**v4**)
= (**v2**•**v3**)∗**v**•(**v1**∗**v4**)

so the action of any vector on (**v1**∗**v2**)•(**v3**∗**v4**) is the same as that on (**v2**•**v3**)∗(**v1**∗**v4**).

Breton: "So it appears that the inner and outer products act together, inner products producing scalars and outer products producing transformations.

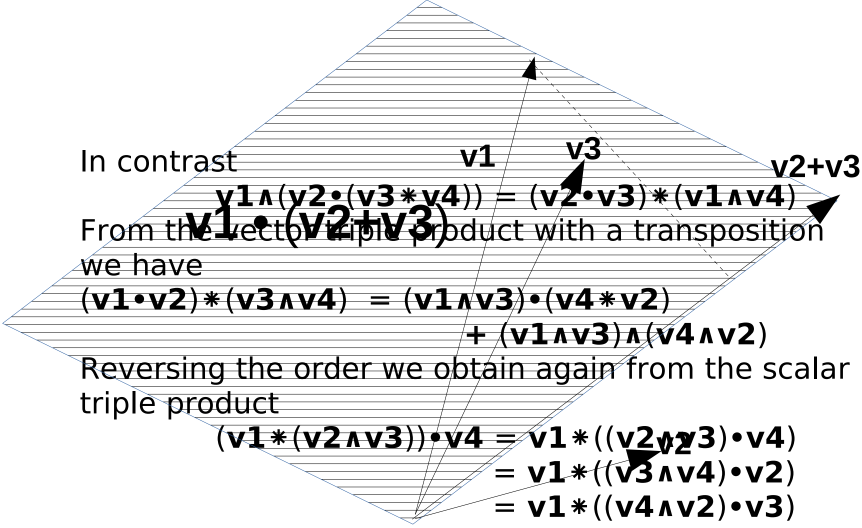
Einstein: "How about the combination {inner, vector, outer}?

Newton: "Not too difficult

v1•((**v2**∧**v3**)∗**v4**) = (**v1**•(**v2**∧**v3**))∗**v4**
= (**v2**•(**v3**∧**v1**))∗**v4**
= (**v3**•(**v1**∧**v2**))∗**v4**
= ((**v1**∧**v2**)•**v3**)∗**v4**
= ((**v2**∧**v3**)•**v1**)∗**v4**
= ((**v3**∧**v1**)•**v2**)∗**v4**
= **v2**•((**v3**∧**v1**)∗**v4**)
= **v3**•((**v1**∧**v2**)∗**v4**)

which are all variations of the triple scalar product as well as.

(**v1**∧**v2**)•(**v3**∗**v4**) .= (**v1**•(**v2**∧**v3**))∗**v4**



In contrast

v1∧(**v2•**(**v3∗v4**)) = (**v2•v3**)∗(**v1∧v4**)

From the vector triple product with a transposition we have

(**v1•v2**)∗(**v3∧v4**) = (**v1∧v3**)•(**v4∗v2**)
+ (**v1∧v3**)∧(**v4∧v2**)

Reversing the order we obtain again from the scalar triple product

(**v1∗**(**v2∧v3**))•**v4** = **v1∗**((**v2∧v3**)•**v4**)
= **v1∗**((**v3∧v4**)•**v2**)
= **v1∗**((**v4∧v2**)•**v3**)
= **v1∗**(**v2•**(**v3∧v4**))
= **v1∗**(**v3•**(**v4∧v2**))
= **v1∗**(**v4**•(**v2∧v3**))

Breton: "The effort to prove the triple product is paying dividends.

Einstein: "The combination {vector ,cross, outer} is missing.

Breton: "For a good reason. The operation

v∧(**v1∗v2**)

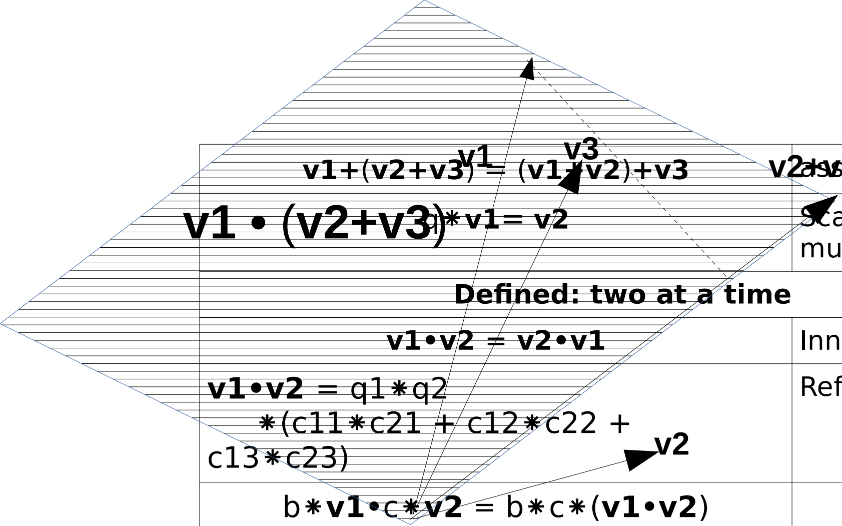
is not defined. It is worthwhile noting that

v1•(**v2∗v3**)–(**v2∗v3**)•**v1** = (**v2∧v3**)**∧v1**,

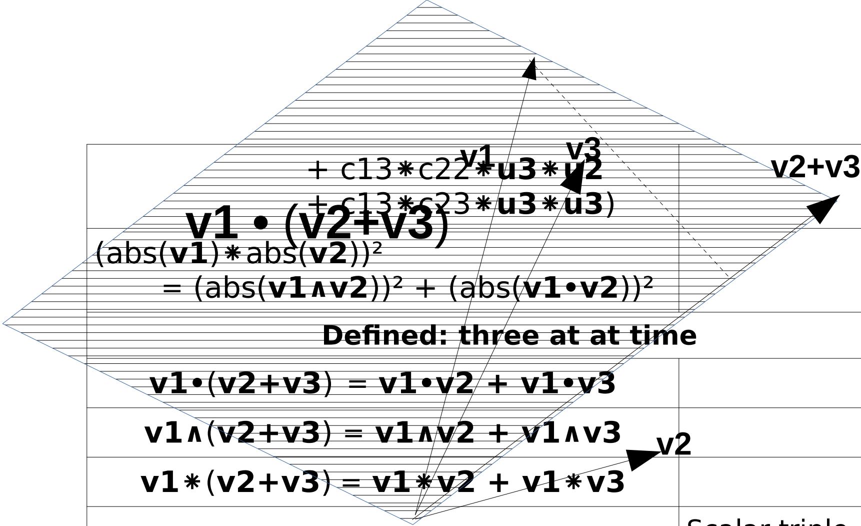
not **0** generally, and that (**v1∧v2**)•(**v3∗v4**) does not equal **v1∧**(**v2•**(**v3∗v4**)).
Now Newton, would you be good enough to put all of these into a table to which we can easily refer.

Newton: "Gladly.

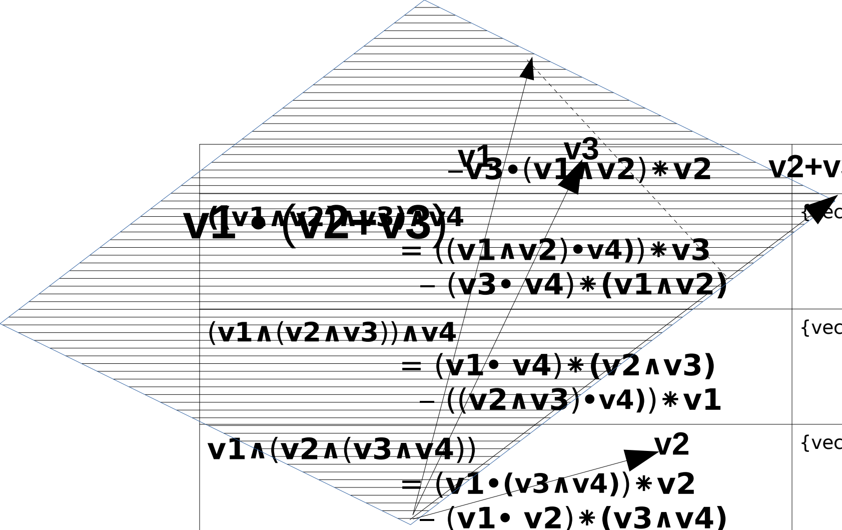
Axiomatic	**Comments**
v1+v2 = v3	closure
v1+v2 = (q1∗c11+ q2∗c21)∗**u1** + (q1∗c12+ q2∗c22)∗**u2** + (q1∗c13+ q2∗c23)∗**u3**	Reference origin



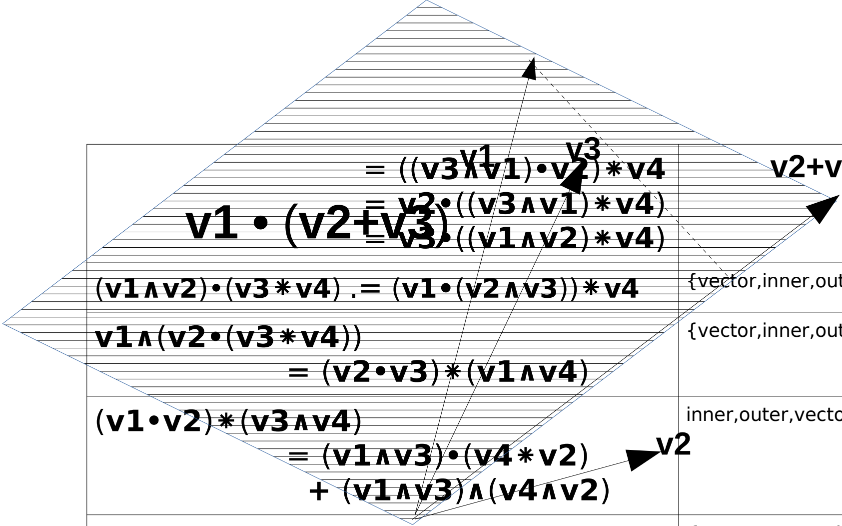
v1+(**v2**+**v3**) = (**v1**+**v2**)+**v3**	association
q∗**v1**= **v2**	Scalar multiplication
Defined: two at a time	
v1•v2 = **v2•v1**	Inner product
v1•v2 = q1∗q2 ∗(c11∗c21 + c12∗c22 + c13∗c23)	Reference origin
b∗**v1**•c∗**v2** = b∗c∗(**v1•v2**)	
abs(**v1•v2**) ≤ abs(**v1**) ∗ abs(**v2**)	
v1∧v2 = –(**v2∧v1**) = ((–**v2**)∧**v1**) = (**v2∧**(–**v1**))	cross product
u1∧u2 ≡ u3 **u2∧u3 ≡ u1** **u3∧u1 ≡ u2**	Reference origin
v1∧v2 = q1∗q2 ∗((c12∗c23 – c13∗c22)∗**u1** + (c13∗c21 – c11∗c23)∗**u2** + (c11∗c22 – c12∗c21)∗**u3**)	Reference origin
v1∧v1 = **0**	
v1•(**v1∧v2**) = **v2**•(**v1∧v2**) = 0	
(b∗**v1**)∧(c∗**v2**) = b∗c∗(**v1∧v2**)	
abs(**v1∧v2**) ≤ abs(**v1**)∗abs(**v2**)	
(b∗**v1**)∗(c∗**v2**) = b∗c∗(**v1∗v2**)	
v1∗v2 = q1∗q2∗**uv1∗uv2**	Reference origin
v1∗v2 = q1∗q2 ∗(c11∗c12∗**u1∗u1** + c11∗c22∗**u1∗u2** + c11∗c23∗**u1∗u3** + c12∗c12∗**u2∗u1** + c12∗c22∗**u2∗u2** + c12∗c23∗**u2∗u3** + c13∗c12∗**u3∗u1**	Reference origin



$+ c13\ast c22\ast\mathbf{u3}\ast\mathbf{u2}$ $+ c13\ast c23\ast\mathbf{u3}\ast\mathbf{u3})$	
$(\mathrm{abs}(\mathbf{v1})\ast\mathrm{abs}(\mathbf{v2}))^2$ $= (\mathrm{abs}(\mathbf{v1}\wedge\mathbf{v2}))^2 + (\mathrm{abs}(\mathbf{v1}\bullet\mathbf{v2}))^2$	
Defined: three at at time	
$\mathbf{v1}\bullet(\mathbf{v2}+\mathbf{v3}) = \mathbf{v1}\bullet\mathbf{v2} + \mathbf{v1}\bullet\mathbf{v3}$	
$\mathbf{v1}\wedge(\mathbf{v2}+\mathbf{v3}) = \mathbf{v1}\wedge\mathbf{v2} + \mathbf{v1}\wedge\mathbf{v3}$	
$\mathbf{v1}\ast(\mathbf{v2}+\mathbf{v3}) = \mathbf{v1}\ast\mathbf{v2} + \mathbf{v1}\ast\mathbf{v3}$	
$\mathbf{v1}\bullet(\mathbf{v2}\wedge\mathbf{v3}) = \mathbf{v2}\bullet(\mathbf{v3}\wedge\mathbf{v1})$ $= \mathbf{v3}\bullet(\mathbf{v1}\wedge\mathbf{v2})$ $= (\mathbf{v1}\wedge\mathbf{v2})\bullet\mathbf{v3}$ $= (\mathbf{v2}\wedge\mathbf{v3})\bullet\mathbf{v1}$ $= (\mathbf{v3}\wedge\mathbf{v1})\bullet\mathbf{v2}$	Scalar triple product
$\mathbf{v1}\bullet(\mathbf{v2}\ast\mathbf{v3}) = (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3}$	transformation
$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) = (\mathbf{v1}\bullet\mathbf{v3})\ast\mathbf{v2}$ $- (\mathbf{v1}\bullet\mathbf{v2})\ast\mathbf{v3}$ $= \mathbf{v3}\bullet(\mathbf{v1}\ast\mathbf{v2})$ $- \mathbf{v3}\ast(\mathbf{v1}\bullet\mathbf{v2})$ $= \mathbf{v1}\bullet(\mathbf{v3}\ast\mathbf{v2} - \mathbf{v2}\ast\mathbf{v3})$	Vector triple product
$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3}) - (\mathbf{v1}\wedge\mathbf{v2})\wedge\mathbf{v3}$ $= \mathbf{v2}\bullet(\mathbf{v3}\ast\mathbf{v1} - \mathbf{v1}\ast\mathbf{v3})$	
$\mathbf{v1}\wedge(\mathbf{v2}\wedge\mathbf{v3})$ $+\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v1})$ $+ \mathbf{v3}\wedge(\mathbf{v1}\wedge\mathbf{v2}) = \mathbf{0}$	
Defined: four at at time	
$(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4})$ $= \mathbf{v1}\bullet(\mathbf{v2}\wedge(\mathbf{v3}\wedge\mathbf{v4}))$ $= ((\mathbf{v1}\bullet\mathbf{v3})\ast(\mathbf{v2}\bullet\mathbf{v4})$ $- (\mathbf{v1}\bullet\mathbf{v2})\ast(\mathbf{v3}\bullet\mathbf{v4})$	{vector,inner,vector}
$(\mathbf{v1}\wedge\mathbf{v2})\bullet(\mathbf{v3}\wedge\mathbf{v4})$ $+ (\mathbf{v1}\wedge\mathbf{v3})\bullet(\mathbf{v4}\wedge\mathbf{v2})$ $+ (\mathbf{v1}\wedge\mathbf{v4})\bullet(\mathbf{v2}\wedge\mathbf{v3}) = 0$	{vector,inner,vector}
$(\mathbf{v1}\wedge\mathbf{v2})\wedge(\mathbf{v3}\wedge\mathbf{v4})$ $= \mathbf{v4}\bullet(\mathbf{v1}\wedge\mathbf{v2})\ast\mathbf{v3}$	{vector,vector,vector}



$-v3\bullet(v1\wedge v2)\ast v2$	
$(v1\wedge(v2\wedge v3))\wedge v4$ $= ((v1\wedge v2)\bullet v4))\ast v3$ $-(v3\bullet v4)\ast(v1\wedge v2)$	{vector,vector,vector}
$(v1\wedge(v2\wedge v3))\wedge v4$ $= (v1\bullet v4)\ast(v2\wedge v3)$ $-((v2\wedge v3)\bullet v4))\ast v1$	{vector,vector,vector}
$v1\wedge(v2\wedge(v3\wedge v4))$ $= (v1\bullet(v3\wedge v4))\ast v2$ $-(v1\bullet v2)\ast(v3\wedge v4)$	{vector,vector,vector}
$v1\wedge((v2\wedge v3)\wedge v4)$ $= (v1\bullet v4)\ast(v2\wedge v3)$ $-(v1\bullet(v2\wedge v3))\ast v4$	{vector,vector,vector}
$(v1\bullet(v2\wedge v4))\ast v3$ $+ ((v2\wedge v3)\bullet v4))\ast v1$ $= (v1\bullet v4)\ast(v2\wedge v3)$ $+ (v1\bullet v3)\ast(v2\wedge v4)$	{inner,vector,outer}
$v1\bullet(v2\bullet v3\ast v4) = (v1\bullet v4)\ast(v2\bullet v3)$	{inner,inner,outer}
$v1\bullet(v2\ast v3)\bullet v4$ $= (v1\bullet v2)\ast(v3\bullet v4)$ $= v1\bullet(v2\ast v4)\bullet v3$ $= v2\bullet(v1\ast v3)\bullet v4$ $= v2\bullet(v1\ast v4)\bullet v3$	{inner,outer,inner}
$(v1\bullet v2)\ast(v3\ast v4)$ $= (v1\bullet(v2\ast v3))\ast v4$ $= v3\ast(v1\bullet v2)\ast v4$ $= (v3\ast v1)\bullet(v2\ast v4)$ $= v3\ast(v2\bullet v1)\ast v4$ $= (v3\ast v2)\bullet(v1\ast v4)$	{inner,outer,outer}
$(v1\ast v2)\bullet(v3\ast v4)$ $= (v2\bullet v3)\ast(v1\ast v4)$	{outer,inner,outer}
$v1\bullet((v2\wedge v3)\ast v4) = (v1\bullet(v2\wedge v3))\ast v4$ $= (v2\bullet(v3\wedge v1))\ast v4$ $= (v3\bullet(v1\wedge v2))\ast v4$ $= ((v1\wedge v2)\bullet v3)\ast v4$ $= ((v2\wedge v3)\bullet v1)\ast v4$	{inner, vector, outer}



$= ((v3 \wedge v1)\bullet v2)\ast v4$ $= v2\bullet((v3 \wedge v1)\ast v4)$ $= v3\bullet((v1 \wedge v2)\ast v4)$	
$(v1 \wedge v2)\bullet(v3 \ast v4)$.= $(v1\bullet(v2 \wedge v3))\ast v4$	{vector,inner,outer}
$v1 \wedge (v2\bullet(v3 \ast v4))$ $= (v2\bullet v3)\ast(v1 \wedge v4)$	{vector,inner,outer}
$(v1\bullet v2)\ast(v3 \wedge v4)$ $= (v1 \wedge v3)\bullet(v4 \ast v2)$ $+ (v1 \wedge v3)\wedge(v4 \wedge v2)$	inner,outer,vector}
$(v1 \ast (v2 \wedge v3))\bullet v4$ $= v1 \ast ((v2 \wedge v3)\bullet v4)$ $= v1 \ast ((v3 \wedge v4)\bullet v2)$ $= v1 \ast ((v4 \wedge v2)\bullet v3)$ $= v1 \ast (v2\bullet(v3 \wedge v4))$ $= v1 \ast (v3\bullet(v4 \wedge v2))$ $= v1 \ast (v4\bullet(v2 \wedge v3))$	{outer,vector,inner}

Breton: "You have constructed a remarkable table, Newton, a veritable armory of intellectual tools. We have climbed much higher on our mountain climb.

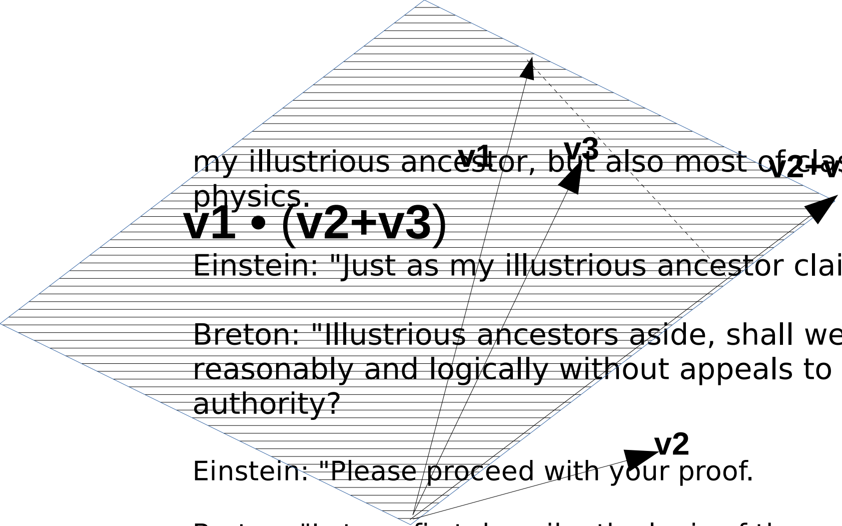
Newton: "How fascinating that each of these vectorial expressions corresponds to a geometrical theorem whose proof might be very difficult indeed.

Einstein: "This wonderful facility arises from the way we defined the origin. The vectorial origin corresponds to the zero of the quotient numbers. It must be special indeed.

Newton: "Just as my illustrious ancestor said and furthermore both named this origin and claimed it as an absolute location.

Breton: "Whatever claim made about a physical origin, I am set to prove that the origin of the set of vectors is completely arbitrary,

Newton: "I shall oppose your reasoning with every ounce of my fiber. Success for you would devastate not only



my illustrious ancestor, but also most of classical physics.

Einstein: "Just as my illustrious ancestor claimed.

Breton: "Illustrious ancestors aside, shall we just proceed reasonably and logically without appeals to previous authority?

Einstein: "Please proceed with your proof.

Breton: "Let me first describe the logic of the proof. Suppose a given origin has been chosen. If ant other vector in the set of vectors can replace the given origin, then the choice of origin is arbitrary.

Newton: "What do you mean by replace?

Breton: "That all the elementarty functions of vectors referred to the initial origin can be reexpressed in terms of the second origin.

Newton: "But the expressions will different.

Breton: "Very likely.

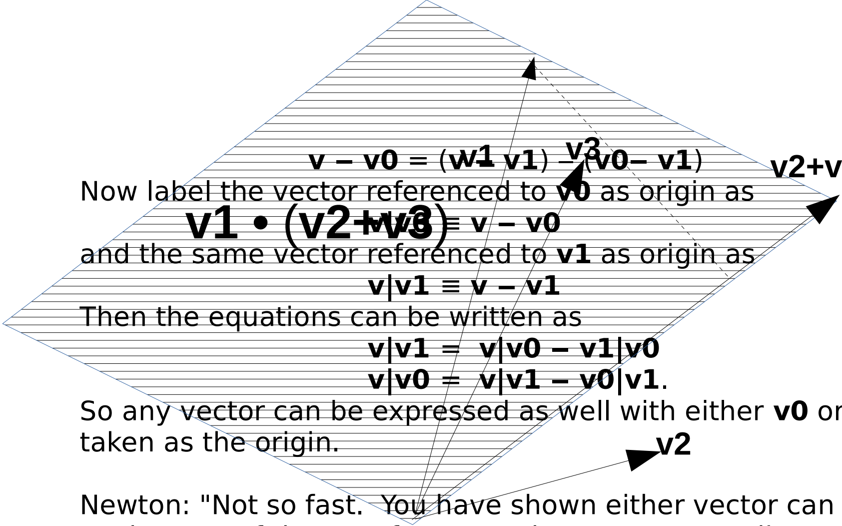
Newton: "Different expressions mean the origins are not arbitrary. One may be preferable to another.

Breton: "Still both expressions are valid. There is no intrinsic reason for choosing one over the other.

Einstein: "Similar to look‒alike functions where two different expressions have the same value. So we will have, if Breton can prove his contention, two different descriptions of the same thing, which is not at all the same as two different descriptions of two different things. Let's get on with the proof.

Breton: "First let me label the given origin **v0**. Secondly, let me take another vector **v1** as a candidate for the new origin. Then for any vector **v**

$$\mathbf{v} - \mathbf{v1} = (\mathbf{v} - \mathbf{v0}) - (\mathbf{v1} - \mathbf{v0})$$



$$\mathbf{v - v0} = (\mathbf{v - v1}) - (\mathbf{v0 - v1})$$

Now label the vector referenced to **v0** as origin as

$$\mathbf{v|v0} \equiv \mathbf{v - v0}$$

and the same vector referenced to **v1** as origin as

$$\mathbf{v|v1} \equiv \mathbf{v - v1}$$

Then the equations can be written as

$$\mathbf{v|v1} = \mathbf{v|v0 - v1|v0}$$
$$\mathbf{v|v0} = \mathbf{v|v1 - v0|v1}.$$

So any vector can be expressed as well with either **v0** or **v1** taken as the origin.

Newton: "Not so fast. You have shown either vector can serve as the zero of the set of vectors, but not necessarily as origin. The origin incorporates an orientation, remember?

Breton: "How could I forget? Let

$$\mathbf{A} = \mathbf{u1{*}ur1 + u2{*}ur2 + u3{*}ur3}$$

where **u1**, **u2**, **u3** are the orientation of **v0** and **ur1**, **ur2**, **ur3** are three mutually orthogonal directions which will serve as the orientation of **v1**.

Then A has an inverse

$$\mathbf{A}^{-1} = \mathbf{ur1{*}u1 + ur2{*}u2 + ur3{*}u3}$$

since $\mathbf{A}\bullet\mathbf{A}^{-1} = \mathbf{I}$, the identity transformation.

Now for any vector **v**

$$(\mathbf{v - v0})\bullet \mathbf{I} = ((\mathbf{v - v1}) - (\mathbf{v0 - v1}))\bullet\mathbf{A}\bullet\mathbf{A}^{-1}$$
$$= (\mathbf{v - v1})\bullet\mathbf{A}\bullet\mathbf{A}^{-1} - (\mathbf{v0 - v1})\bullet\mathbf{A}\bullet\mathbf{A}^{-1}$$

Now let

$$\mathbf{v|v1} = (\mathbf{v - v1})\bullet\mathbf{A}$$

to indicate both a change in position and reorientation.

Then

$$\mathbf{v|v1} = (\mathbf{v|v0 - v1|v0})\bullet\mathbf{A}$$

for **v1** as origin, and

$$\mathbf{v|v0} = (\mathbf{v|v1 - v0|v1})\bullet\mathbf{A}^{-1}$$

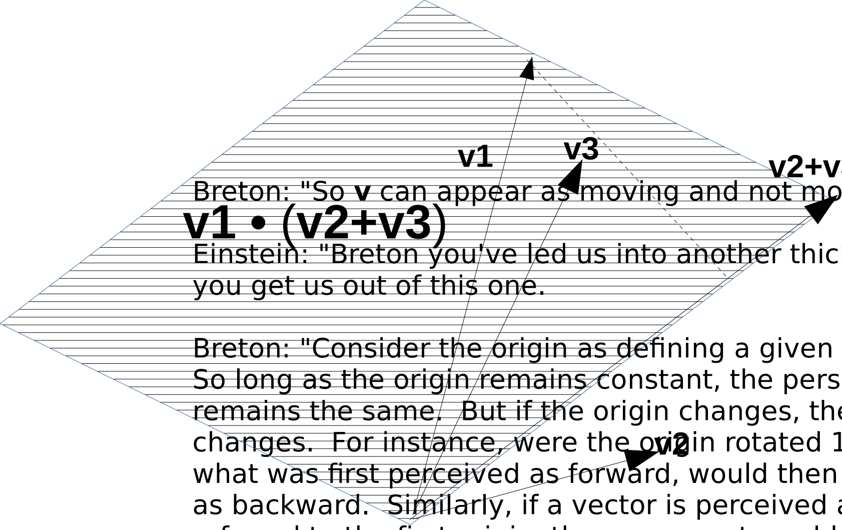
for **v0** as origin.

Thus **v1** is as suitable to serve as reference as **v0**, both for locations and orientations, and thus as origin for any vector **v**.

Einstein: "Exactly. No location is absolute; any location is only relative to the origin. But suppose the origin itself is moving.

Breton: "How would you know it is moving?

Newton: "If the vector **v** and the origin were both moving identically, then it would appear that **v** is not moving.



Breton: "So **v** can appear as moving and not moving.

Einstein: "Breton you've led us into another thicket. How will you get us out of this one.

Breton: "Consider the origin as defining a given perspective. So long as the origin remains constant, the perspective remains the same. But if the origin changes, the perspective changes. For instance, were the origin rotated 180 degrees, what was first perceived as forward, would then be perceived as backward. Similarly, if a vector is perceived as moving referred to the first origin, the movement would appear different referred to a different origin. If the second origin were moving, the perspective might change enough to make the vector appear to have stopped moving.

Einstein: "So there is no true location or true motion?

Breton: "No absolutely true location or motion exists physically, only location and motion referred to an origin.

Einstein: "That is all location and motion are relative.

Breton: "It is not their existence which is relative, but only our perception of them.

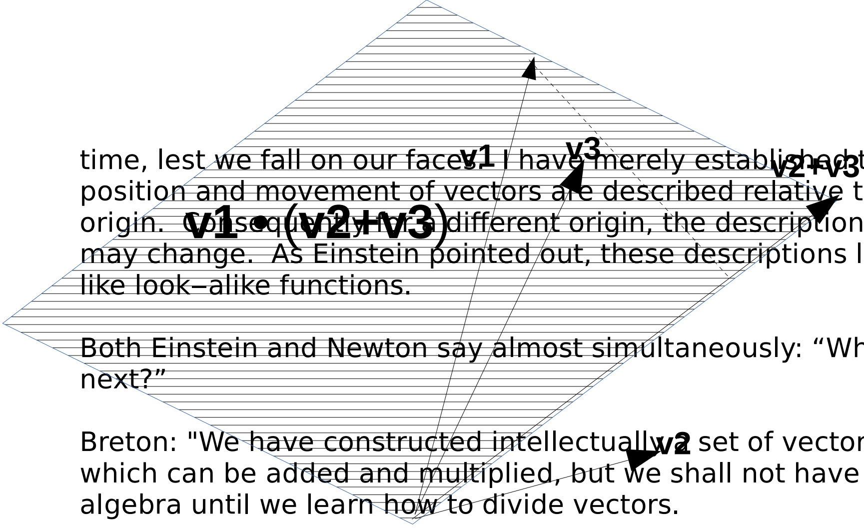
Newton: "I disagree vehemently. When my illustrious ancestor sought to understand the forces operating in the solar system, he used the perspective of the motion relative to the sun as origin. On this basis he formulate the axioms of his *Principia.* If these axioms are absolutely true, and let me add they result in his definition of gravity, then the sun must be an absolute location.

Breton: "And if the sun is not an absolute location, then his axioms are not absolutely true.

Newton: "Then classical mechanics is not absolutely true.

Einstein: "Of course, just as my illustrious ancestor said.

Breton: "Well now, we have departed from considering vector sets and their proposed algebra. Let us take one step at a



time, lest we fall on our faces. I have merely established that position and movement of vectors are described relative to an origin. Consequently, for a different origin, the description may change. As Einstein pointed out, these descriptions look like look‒alike functions.

Both Einstein and Newton say almost simultaneously: “What next?”

Breton: "We have constructed intellectually a set of vectors which can be added and multiplied, but we shall not have an algebra until we learn how to divide vectors.

Einstein: "Ask any mathematician, vectors cannot be divided.

Breton: "We found that multiplication in the set of vectors needed to be defined. So let us try to define division in the set of vectors without reference to opinion of others which may be erroneous.

Newton: "How?

Breton: "Remember in the set of quotient numbers, Q, for every quotient number, q, except 0. we could find another quotient number q1 such that q ∗ q1 = 1. Thus we could define 1/q ≡ q1 and so we were came to define division in Q using reciprocal quotient numbers.
Now any vector **v** = q(**v**)∗**uv**. We already have a reciprocal for q(**v**) so we only need deal with **uv**.

Newton: "And we already know **uv•uv**. = 1. So for any vector **v**

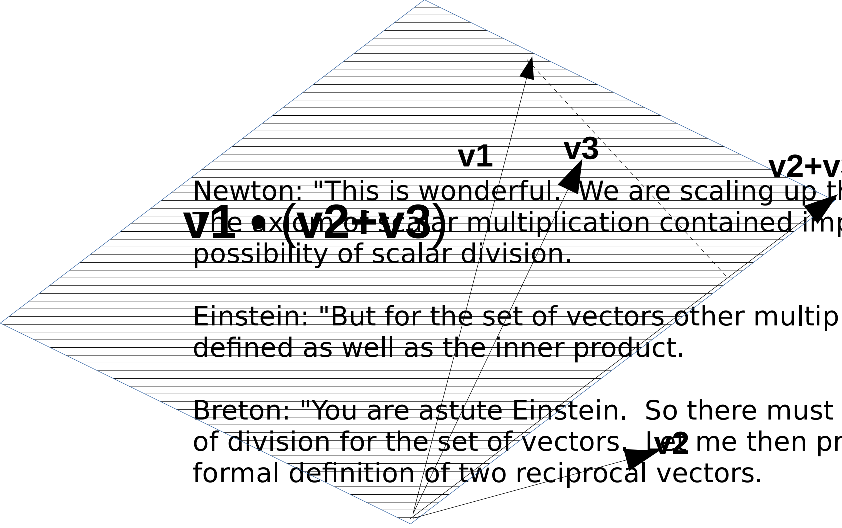
$$\mathbf{v} \bullet((1/q(\mathbf{v}))*\mathbf{uv}) = q(\mathbf{v})*\mathbf{uv}\bullet((1/q(\mathbf{v}))*\mathbf{uv}) = 1.$$

Breton: "By your leave let us label the vector ((1/q(**v**))∗**uv**)) as **qd**(**v**).
So you see dear Einstein, division in the set of vectors is not only possible, but even easy. For every vector **v** there exists a reciprocal vector **qd**(**v**) such that

$$\mathbf{v}\bullet\mathbf{qd}(\mathbf{v}) = 1.$$

Einstein: "Except for **0**.

Breton: "Similar to the quotient numbers.



Newton: "This is wonderful. We are scaling up the mountain. The axiom of scalar multiplication contained implicitly the possibility of scalar division.

Einstein: "But for the set of vectors other multiplications are defined as well as the inner product.

Breton: "You are astute Einstein. So there must be other kinds of division for the set of vectors. Let me then propose a formal definition of two reciprocal vectors.

> **Definition** (reciprocal vectors)
> Given
> $$\mathbf{v} = \mathrm{q}(\mathbf{v})*\mathbf{uv} = \mathrm{q}(\mathbf{v})*(\mathrm{c1}*\mathbf{u1}+\mathrm{c2}*\mathbf{u2}+\mathrm{c3}*\mathbf{u3})$$
> then
> $$\mathbf{qd}(\mathbf{v}) \equiv \mathbf{uv}/\mathrm{q}(\mathbf{v}) = (\mathrm{c1}*\mathbf{u1}+\mathrm{c2}*\mathbf{u2}+\mathrm{c3}*\mathbf{u3})/\mathrm{q}(\mathbf{v})$$
> is called the **directional reciprocal vector** of **v**,
> and
> $$\mathbf{q}(\mathbf{v}) \equiv \mathbf{q}(\mathbf{uv})/\mathrm{q}(\mathbf{v}) = (\mathbf{u1}/\mathrm{c1}+\mathbf{u2}/\mathrm{c2}+\mathbf{u3}/\mathrm{c3})/\mathrm{q}(\mathbf{v})$$
> is called the **general reciprocal vector** of **v**,
> end of definition

Notice that

$$\mathbf{v}\bullet\mathbf{qd}(\mathbf{v}) = 1$$

while

$$\mathbf{v}\bullet\mathbf{q}(\mathbf{v}) = 3.$$

Newton: "Are there others?

Solutions of vector algebraic equations

Breton: "Yes but this will do for now. And now that we have assembled an algebra for our set of vectors, let us solve a few algebraic equations. What is the solution for a vector **x** where

$$\mathbf{x}\bullet\mathbf{v1} = \mathrm{q1}$$

and **v1** is a given vector and q1 is a given quotient number?.

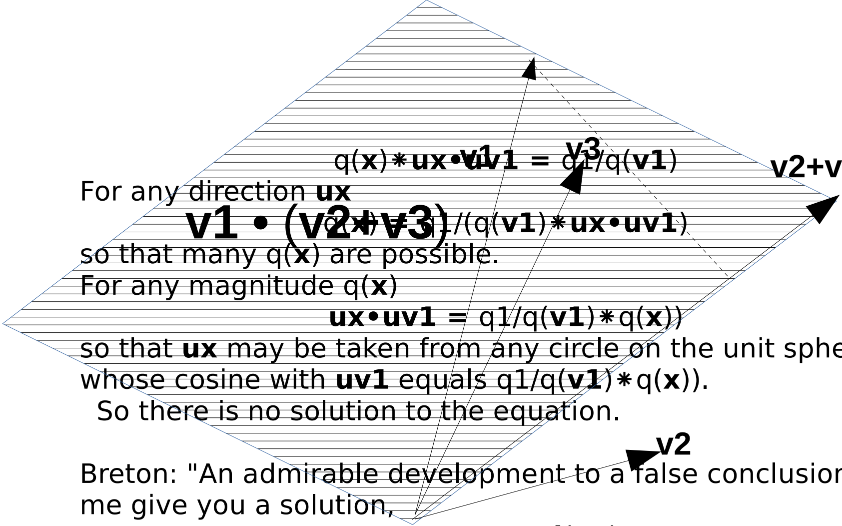
Einstein: "Let's break out the equation a little more. Say

$$\mathbf{x} = \mathrm{q}(\mathbf{x})*\mathbf{ux}$$

and

$$\mathbf{v1} = \mathrm{q}(\mathbf{v1})*\mathbf{uv1}$$

Then



$$q(\mathbf{x})*\mathbf{ux}\bullet\mathbf{uv1} = q1/q(\mathbf{v1})$$

For any direction **ux**

$$q(\mathbf{x}) = q1/(q(\mathbf{v1})*\mathbf{ux}\bullet\mathbf{uv1})$$

so that many q(**x**) are possible.
For any magnitude q(**x**)

$$\mathbf{ux}\bullet\mathbf{uv1} = q1/q(\mathbf{v1})*q(\mathbf{x}))$$

so that **ux** may be taken from any circle on the unit sphere whose cosine with **uv1** equals q1/q(**v1**)∗q(**x**)).
So there is no solution to the equation.

Breton: "An admirable development to a false conclusion. Let me give you a solution,

$$\mathbf{x1} = q1*\mathbf{qd}(\mathbf{v1})$$

since q1∗**qd**(**v1**)•**v1** = q1∗1 = q1.

Newton: "So Einstein's argument shows not there are no solutions, but there are a great many with differing directions and magnitudes.

Breton: "The truth of the matter is sprouting. We can go further. Let **x1** and **x2** be two different solutions. Then

$$(\mathbf{x1} - \mathbf{x2})\bullet\mathbf{v1} = q1 - q1 = 0$$

so the difference between any two solution vectors is orthogonal to **v1**. Consequently, the entire set of solutions describes a plane orthogonal to **v1**
In particular for any solution vector let

$$\mathbf{x} = q1*\mathbf{qd}(\mathbf{v1}) = q*\mathbf{un}$$

for some quotient number q and **un** a direction orthogonal to **v1**. Then the entire set of solutions may be written as

$$\{\mathbf{x}|\mathbf{x} = q1*\mathbf{qd}(\mathbf{v1}) + q*\mathbf{un}\}.$$

Einstein: "What's so special about the reciprocal vector **qd**(**v1**) for a solution?

Breton: "As you have shown, the magnitude of any solution can be written as

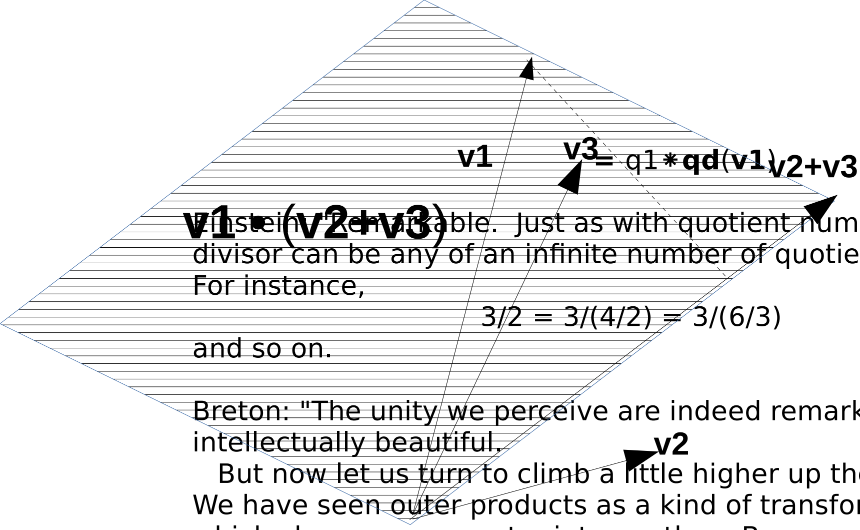
$$q(\mathbf{x}) = q1/(q(\mathbf{v1})*\mathbf{ux}\bullet\mathbf{uv1})$$

so the minimum magnitude is that which maximizes **ux•uv1**.

Newton: "Which is **ux** = **uv1**!

Breton: "Exactly. Then the unique solution with minimum magnitude is

$$\mathbf{x} = q(\mathbf{x})*\mathbf{ux} = q1*\mathbf{uv1}/q(\mathbf{v1})$$



Einstein: "Remarkable. Just as with quotient numbers, a divisor can be any of an infinite number of quotient numbers. For instance,

$$3/2 = 3/(4/2) = 3/(6/3)$$

and so on.

Breton: "The unity we perceive are indeed remarkable, and intellectually beautiful.

But now let us turn to climb a little higher up the mountain. We have seen outer products as a kind of transformation which changes one vector into another. Because of its utility let me now consider transformations more generally as matrices.

Einstein: "A new word. Please define what you mean.

Breton: "Of course. Here is a definition.

> **Definition** (matrix)
> Given
> three vectors, **v1**, **v2**, **v3**.
> And origin designated by **u1**, **u2**, **u3**
> then a matrix **A** is defined as
> $$\mathbf{A} \equiv \mathbf{u1} * \mathbf{v1} + \mathbf{u2} * \mathbf{v2} + \mathbf{u3} * \mathbf{v3}$$
> end of definition

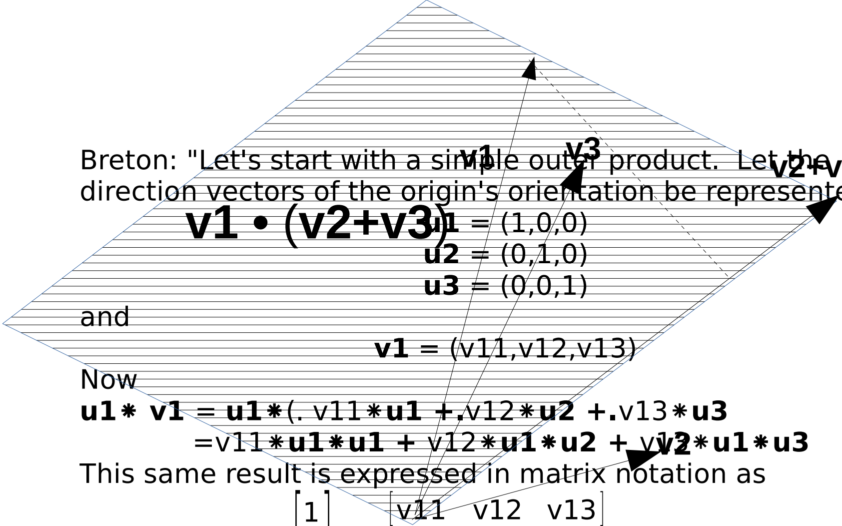
The matrix can be represented as an ordered array of vectors, as follows. For

$$\mathbf{v1} = v11*\mathbf{u1} + .v12*\mathbf{u2} + .v13*\mathbf{u3}$$
$$\mathbf{v2} = v21*\mathbf{u1} + .v22*\mathbf{u2} + .v23*\mathbf{u3}$$
$$\mathbf{v3} = v31*\mathbf{u1} + .v32*\mathbf{u2} + .v33*\mathbf{u3}$$

$$\mathbf{A} = \begin{bmatrix} v11 & v12 & v13 \\ v21 & v22 & v23 \\ v31 & v32 & v33 \end{bmatrix}$$

You can see that the matrix is composed of nine elements arranged in three rows and three columns and implies a given orientation.

Einstein: "How does the matrix fit with the definition?



Breton: "Let's start with a simple outer product. Let the direction vectors of the origin's orientation be represented as

u1 = (1,0,0)
u2 = (0,1,0)
u3 = (0,0,1)

and

v1 = (v11,v12,v13)

Now
u1∗ v1 = **u1∗**(. v11∗**u1 +.**v12∗**u2 +.**v13∗**u3**
=v11∗**u1∗u1 +** v12∗**u1∗u2 +** v13∗**u1∗u3**

This same result is expressed in matrix notation as

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}v11 & v12 & v13\end{bmatrix}$$

where the first element of the column vector , 1, is multiplied by each member of the horizontal vector to form the topmost row of the matrix, the second element of the column vector, 0, is multiplied by each member of the horizontal vector to form the middle row of the matrix , and the third element of the column vector, 0, is multiplied by each member of the horizontal vector to form the bottom row of the matrix

The result becomes

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}v11 & v12 & v13\end{bmatrix} = \begin{bmatrix}v11 & v12 & v13\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

You can see that **u1∗u1** corresponds to position of the first row and first column; that **u1∗u2** corresponds to position of the first row and second column; that **u1∗u3** corresponds to position of the first row and the third column.

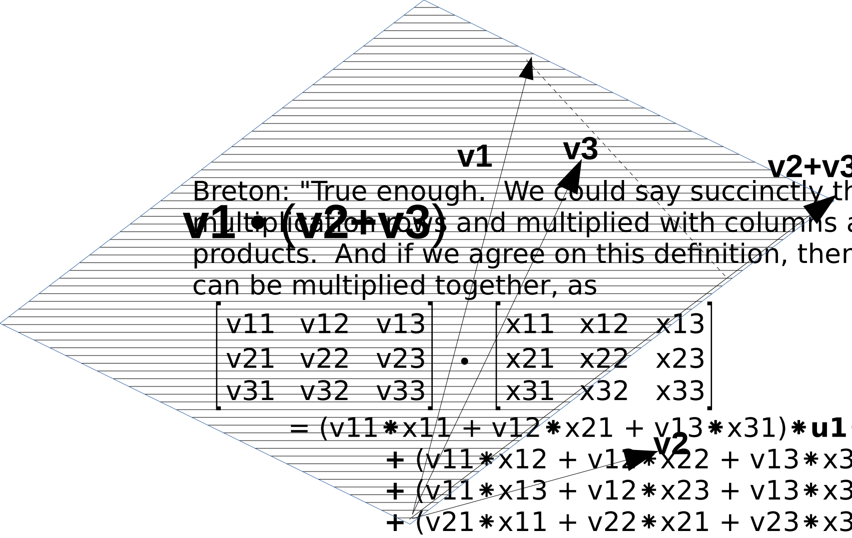
Einstein: "What good is all this bookkeeping for?

Breton: "It does seem complicated, but it matches the complicated process of vector multiplication well. For instance the inner product of two vectors **v1•v2** can be expressed as

$$\begin{bmatrix}v11 & v12 & v13\end{bmatrix} \bullet \begin{bmatrix}v21\\v22\\v23\end{bmatrix} \equiv v11*v21+v12*v22+v13*v23$$

So this matrix multiplication comprehends both inner and outer vector multiplications.

Newton: "You have added to the above description by defining matrix multiplication to resemble inner products.



Breton: "True enough. We could say succinctly that in matrix multiplication rows and multiplied with columns as inner products. And if we agree on this definition, then two matrices can be multiplied together, as

$$\begin{bmatrix} v11 & v12 & v13 \\ v21 & v22 & v23 \\ v31 & v32 & v33 \end{bmatrix} \bullet \begin{bmatrix} x11 & x12 & x13 \\ x21 & x22 & x23 \\ x31 & x32 & x33 \end{bmatrix}$$

$$\begin{aligned} &= (v11*x11 + v12*x21 + v13*x31)*\mathbf{u1}*\mathbf{u1} \\ &\quad \mathbf{+}\ (v11*x12 + v12*x22 + v13*x32)*\mathbf{u1}*\mathbf{u2} \\ &\quad \mathbf{+}\ (v11*x13 + v12*x23 + v13*x33)*\mathbf{u1}*\mathbf{u3} \\ &\quad \mathbf{+}\ (v21*x11 + v22*x21 + v23*x31)*\mathbf{u2}*\mathbf{u1} \\ &\quad \mathbf{+}\ (v21*x12 + v22*x22 + v23*x32)*\mathbf{u2}*\mathbf{u2} \\ &\quad \mathbf{+}\ (v21*x13 + v22*x23 + v23*x33)*\mathbf{u2}*\mathbf{u3} \\ &\quad \mathbf{+}\ (v31*x11 + v32*x21 + v33*x31)*\mathbf{u3}*\mathbf{u1} \\ &\quad \mathbf{+}\ (v31*x12 + v32*x22 + v33*x32)*\mathbf{u3}*\mathbf{u2} \\ &\quad \mathbf{+}\ (v31*x13 + v32*x23 + v33*x33)*\mathbf{u3}*\mathbf{u3} \end{aligned}$$

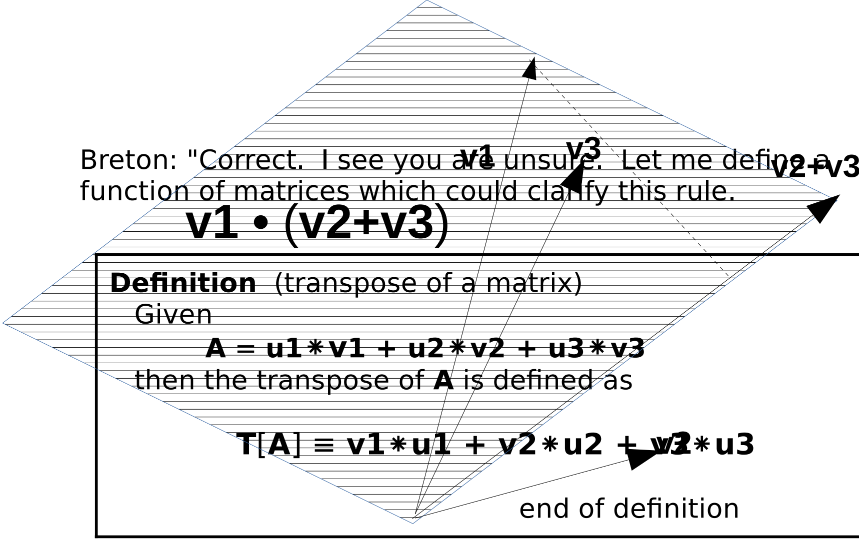
Einstein: "If this defines matrix multiplication, then it is ambiguous! A vector in this notation can be either vertical or horizontal. How can that be?

Breton: "As usual you are astute, my dear Einstein. We must make a choice so let us choose horizontal for a vector. Then **v•A** is another horizontal vector, and so a legitimate operation. But **A•v** is not a meaningful symbol if **v** is taken as a horizontal vector. If however **v** is taken as a vertical vector, the operation results in a vertical vector and so is not a legitimate operation by itself. However, **v1•A•v2** becomes a horizontal vector multiplied by a vertical vector which is an inner product and so legitimate.

Newton: "Now you have gotten us into a fine pickle. Sometimes the vector is horizontal and sometimes vertical.

Breton: "Not really We adopt the rule that a vector is represented horizontally usually but not always. When it follows a matrix it will be presented vertically.

Einstein: "So its position relative to a matrix determines if the vector is represented horizontally or vertically.



Breton: "Correct. I see you are unsure. Let me define a function of matrices which could clarify this rule.

Definition (transpose of a matrix)
Given

$$\mathbf{A} = \mathbf{u1} * \mathbf{v1} + \mathbf{u2} * \mathbf{v2} + \mathbf{u3} * \mathbf{v3}$$

then the transpose of **A** is defined as

$$\mathbf{T}[\mathbf{A}] \equiv \mathbf{v1} * \mathbf{u1} + \mathbf{v2} * \mathbf{u2} + \mathbf{v3} * \mathbf{u3}$$

end of definition

If

$$\mathbf{A} = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$$

then

$$\mathbf{T}[\mathbf{A}] = \begin{bmatrix} a11 & a21 & a33 \\ a12 & a22 & a32 \\ a13 & a23 & a33 \end{bmatrix}$$

Newton: "So the transpose simply exchanges rows and columns.

Breton: "Exactly. So we can apply the transpose notation to horizontal vectors to create a vertical vector. Then had I made all vectors horizontal what I have written above as **A•v** would be written as **A•T**[**v**]. But this is an unnecessary complication. Post vector multiplication can only be meaningful as a vertical vector.

Einstein: "But the transpose notation is still useful?

Breton: "Of course. To signify outer products we would write

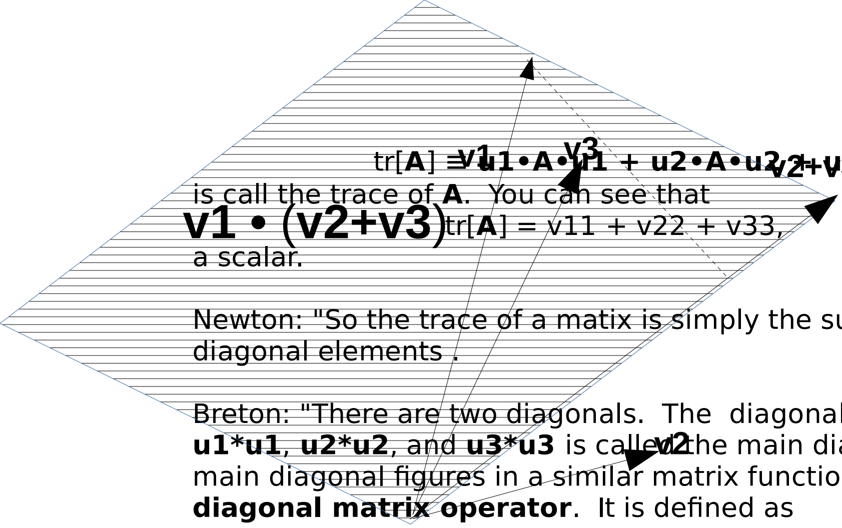
T[**v1**]**• v2** = **v1*v2**

while inner products **v1• v2** would have to be

v1•T[**v2**]

Newton: "So we can formulate functions of matrices as well as of vectors.

Breton: "Yes indeed. Let me define a few for you.



$$\mathrm{tr}[\mathbf{A}] \equiv \mathbf{u1 \bullet A \bullet u1 + u2 \bullet A \bullet u2 + u3 \bullet A \bullet u3}$$

is call the trace of **A**. You can see that

$$\mathrm{tr}[\mathbf{A}] = v11 + v22 + v33,$$

a scalar.

Newton: "So the trace of a matix is simply the sum of its diagonal elements .

Breton: "There are two diagonals. The diagonal with locations **u1*u1**, **u2*u2**, and **u3*u3** is called the main diagonal. The main diagonal figures in a similar matrix function called the **diagonal matrix operator**. It is defined as

$$\mathbf{g}[\mathbf{A}] \equiv v11 \ast \mathbf{u1} + v22 \ast \mathbf{u2} + v33 \ast \mathbf{u3}$$

Newton: "So **g**[**A**] transforms a matrix into a vector specified by its main diagonal. But why do you call it an *operator*.

Breton: " Remember in tp1.1 we said an operator is “a general term for a process applied to a set”. Here we have an operator as a function of a matrix. Let me define another such operator,

$$\mathbf{c}[\mathbf{A}] \equiv (v23-v32) \ast \mathbf{u1} + (v31-v13) \ast \mathbf{u2} + (v12-v21) \ast \mathbf{u3}$$

which is called the **curl matrix operator**. This operator also transforms the matrix into a vector but uses only off-diagonal elements.

Newton: "Which reminds me of vector multiplication.

Breton: "You have good intuitions, Newton. The curl matrix operator, however, operates on a matrix to yield a vector, a process which differs from cross multiplication. Let me introduce you to another matrix operator which is defined in terms now familiar to us

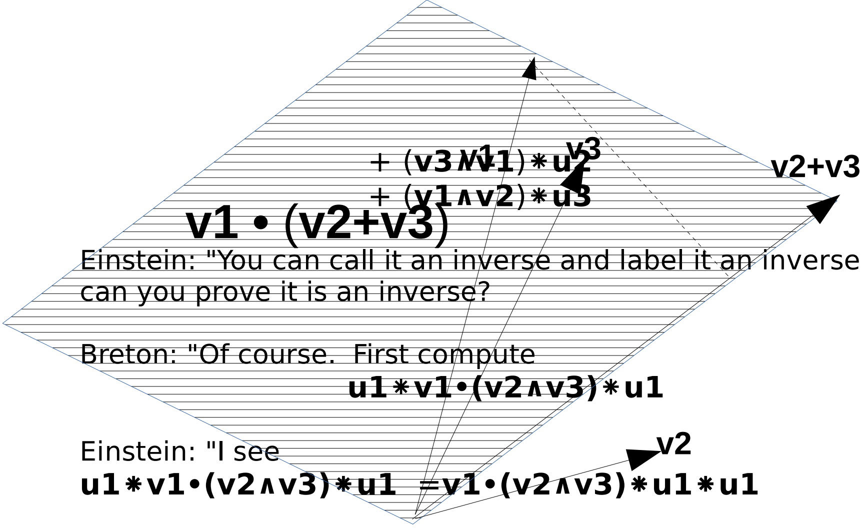
$$\det[\mathbf{A}] \equiv \mathbf{v1} \bullet (\mathbf{v2} \wedge \mathbf{v3}) = \mathbf{v2} \bullet (\mathbf{v3} \wedge \mathbf{v1}) = \mathbf{v3} \bullet (\mathbf{v1} \wedge \mathbf{v2})$$

is called the determinant of a matrix. As you can see it is simply the scalar triple product of its constituent vectors.

Newton: "So the determinant is like the “volume” of a matrix.

Breton: "Something like, but just remember the triple product can also be a negative number. The determinant figures into another operator, the inverse matrix of .**A.**

$$\det[\mathbf{A}] \ast \mathbf{A}^{-1} = (\mathbf{v2} \wedge \mathbf{v3}) \ast \mathbf{u1}$$



+ (**v3∧v1**)∗**u2**
+ (**v1∧v2**)∗**u3**

Einstein: "You can call it an inverse and label it an inverse, but can you prove it is an inverse?

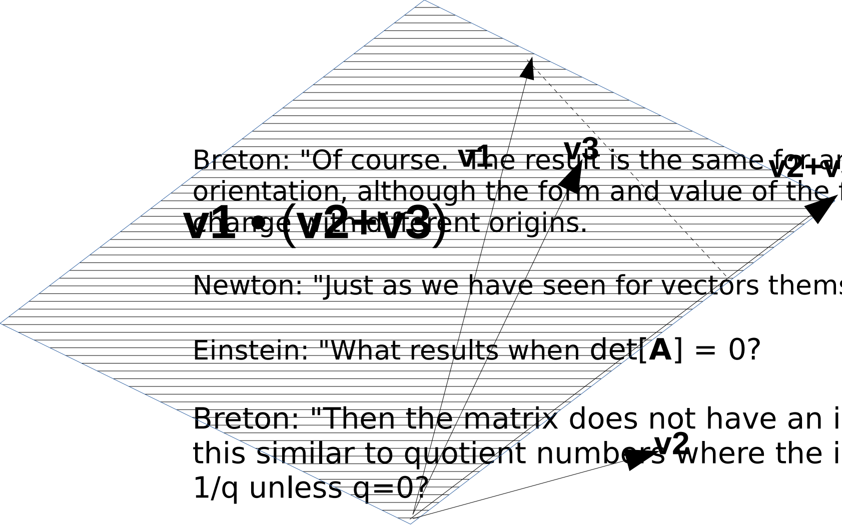
Breton: "Of course. First compute
u1∗v1•(v2∧v3)∗u1

Einstein: "I see
u1∗v1•(v2∧v3)∗u1 =v1•(v2∧v3)∗u1∗u1

Breton: "Which is det[**A**]∗**u1∗u1**. So
A•**A**$^{-1}$ = (**u1∗v1 + u2∗v2 +u3∗v3**)
•((**v2∧v3**)∗**u1**
+ (**v3∧v1**)∗**u2**
+ (**v1∧v2**)∗**u3/**det[**A**]
= (**u1∗v1**)•((**v2∧v3)∗u1**
+ (**v3∧v1**)∗**u2**
+ (**v1∧v2**)∗**u3/**det[**A**]
+ (**u2∗v2**)•((**v2∧v3)∗u1**
+ (**v3∧v1**)∗**u2**
+ (**v1∧v2**)∗**u3/**det[**A**]
+ (**u3∗v3**)•((**v2∧v3)∗u1**
+ (**v3∧v1**)∗**u2**
+ (**v1∧v2**)∗**u3/**det[**A**]
= (**u1∗v1**)•((**v2∧v3)∗u1/**det[**A**]
+ (**u2∗v2**)•((**v3∧v1**)∗**u2/**det[**A**]
+ (**u3∗v3**)•((**v1∧v2**)∗**u3/**det[**A**]
= (**u1∗**det[**A**]∗**u1/**det[**A**]
+ (**u2∗**det[**A**]∗**u2/**det[**A**]
+ (**u3∗**det[**A**]∗**u3/**det[**A**]
= **u1∗u1 + u2∗u2 + u3∗u3**
= **I**
which is the identity transformation.

Newton: "The proof builds on that remarkable property of the scalar triple product. We are building well.

Einstein: "Your proof depends on the arbitrary orientation of the origin.



Breton: "Of course. The result is the same for any given orientation, although the form and value of the factors may change with different origins.

Newton: "Just as we have seen for vectors themselves.

Einstein: "What results when det[**A**] = 0?

Breton: "Then the matrix does not have an inverse. Isn't this similar to quotient numbers where the inverse of q is 1/q unless q=0?

Einstein: "Let's step back a little. We aimed at trying to define an algebra for the set of vectors. Now it seems we have not only accomplished that, but also defined an algebra for matrices. Breton, write the operations for matrix algebra explicitly.

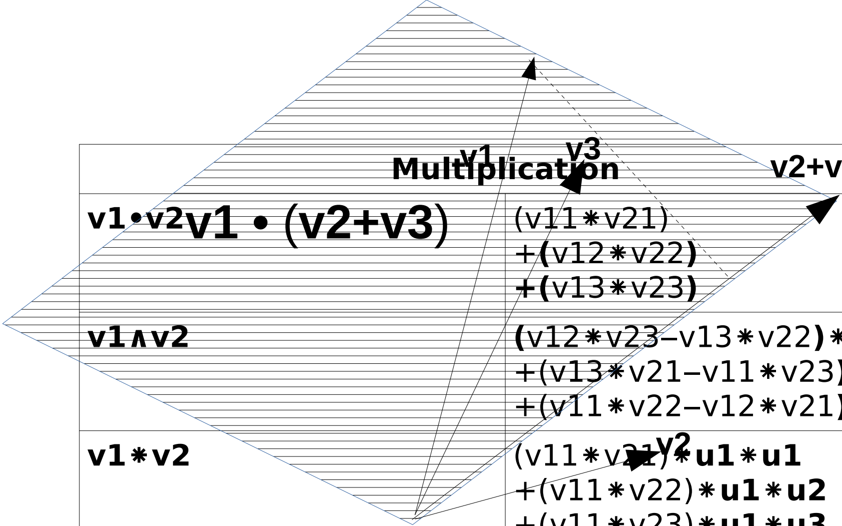
Breton: "Better still let me construct a table showing the algebra of both vectors and matrices. The symbols of the table are defined as

v1 ≡ v1∗**uv1**
v2 ≡ v2∗**uv2**

With the origin taken as reference these vectors are further specified as

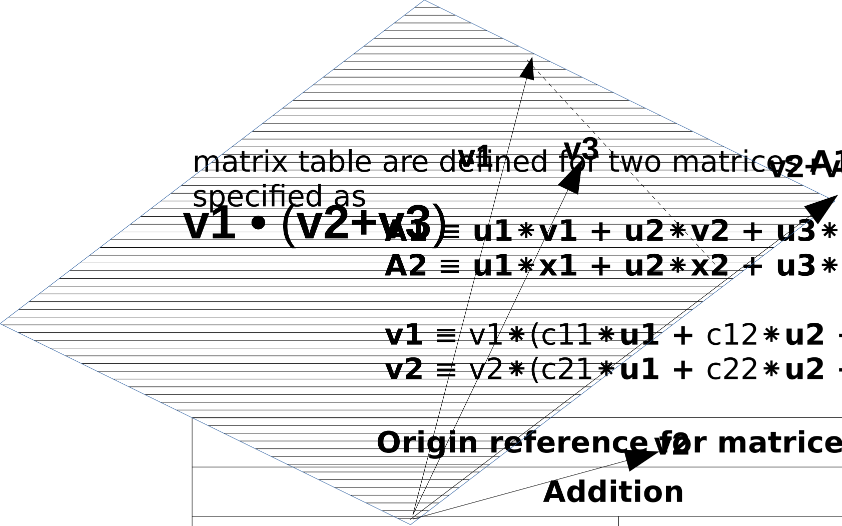
v1 ≡ v11∗**u1** + v12∗**u2** + v13∗**u3**)
v1 ≡ v1∗(c11∗**u1** + c12∗**u2** + c13∗**u3**)
v2 ≡ v21∗**u1** + v22∗**u2** + v23∗**u3**)
v2 ≡ v2∗(c21∗**u1** + c22∗**u2** + c23∗**u3**)

Origin reference for vectors	
Addition	
v1+v2	(v11+v21)∗**u1** **(**v12+v22**)**∗**u2** **(**v13+v23**)**∗**u3**
Subtraction	
v1−v2	(v11−v21)∗**u1** **(**v12−v22**)**∗**u2** **(**v13−v23**)**∗**u3**



Multiplication	
v1•v2	(v11∗v21) +(v12∗v22) +(v13∗v23)
v1∧v2	(v12∗v23−v13∗v22)∗**u1** +(v13∗v21−v11∗v23)∗**u2** +(v11∗v22−v12∗v21)∗**u3**
v1∗v2	(v11∗v21)∗**u1∗u1** +(v11∗v22)∗**u1∗u2** +(v11∗v23)∗**u1∗u3** +(v12∗v21)∗**u2∗u1** +(v12∗v22)∗**u2∗u2** +(v12∗v23)∗**u2∗u3** +(v13∗v21)∗**u3∗u1** +(v13∗v22)∗**u3∗u2** +(v13∗v23)∗**u3∗u3**
Division	
v1•qd(v2)	(v1/v2) ∗(c11∗c21 +c12∗c22 +c13∗c23)
v1∧q(v2)	(v1/v2) ∗((c12/c23−c13/c22)∗**u1** +(c13/c21−c11/c23)∗**u2** +(c11/c22−c12/c21)∗**u3**)
v1∗qd(v2)	(v1/v2) ∗(c11∗c21)∗**u1∗u1** +(c11∗c22)∗**u1∗u2** +(c11∗c23)∗**u1∗u3** +(c12∗c21)∗**u2∗u1** +(c12∗c22)∗**u2∗u2** +(c12∗c23)∗**u2∗u3** +(c13∗c21)∗**u3∗u1** +(c13∗c22)∗**u3∗u2** +(c13∗c23)∗**u3∗u3**

With the origin taken as reference, the symbols for the

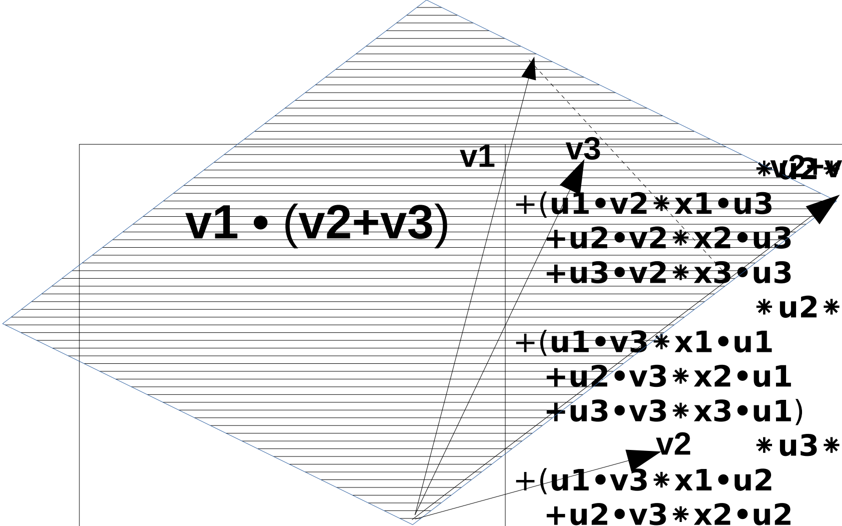


matrix table are defined for two matrices **A1** and **A2** specified as

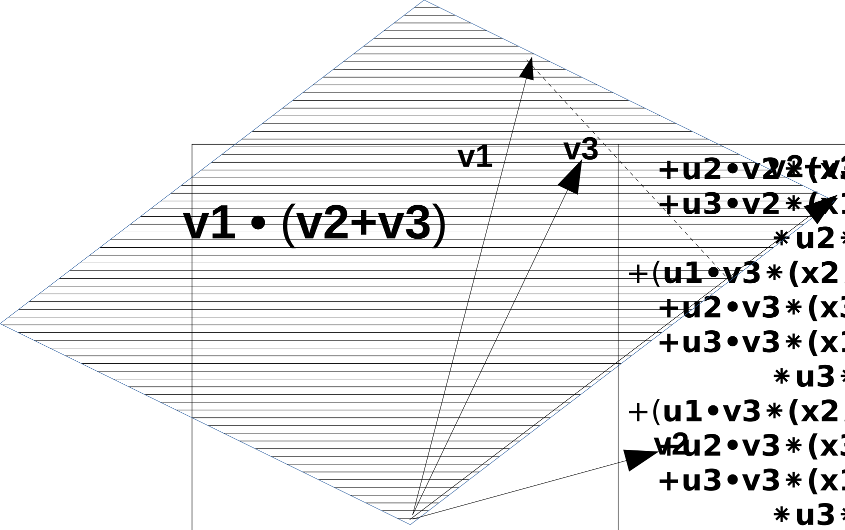
A1 ≡ u1∗v1 + u2∗v2 + u3∗v3
A2 ≡ u1∗x1 + u2∗x2 + u3∗x3

v1 ≡ v1∗(c11∗**u1** + c12∗**u2** + c13∗**u3**)
v2 ≡ v2∗(c21∗**u1** + c22∗**u2** + c23∗**u3**)

Origin reference for matrices	
Addition	
A1+A2	**u1∗(v1+x1)** **+ u2∗(v2+x2)** **+ u3∗(v3+x3)**
Subtraction	
A1−A2	**u1∗(v1−x1)** **+ u2∗(v2−x2)** **+ u3∗(v3−x3)**
Multiplication	
A1•A2	(**u1•v1∗x1•u1** **+u2•v1∗x2•u1** **+u3•v1∗x3•u1**) **∗u1∗u1** +(**u1•v1∗x1•u2** **+u2•v1∗x2•u2** **+u3•v1∗x3•u2** **∗u1∗u2** +(**u1•v1∗x1•u3** **+u2•v1∗x2•u3** **+u3•v1∗x3•u3** **∗u1∗u3** +(**u1•v2∗x1•u1** **+u2•v2∗x2•u1** **+u3•v2∗x3•u1**) **∗u2∗u1** +(**u1•v2∗x1•u2** **+u2•v2∗x2•u2** **+u3•v2∗x3•u2**



	∗u2∗u2 +(**u1•v2∗x1•u3** **+u2•v2∗x2•u3** **+u3•v2∗x3•u3** **∗u2∗u3** +(**u1•v3∗x1•u1** **+u2•v3∗x2•u1** **+u3•v3∗x3•u1**) **∗u3∗u1** +(**u1•v3∗x1•u2** **+u2•v3∗x2•u2** **+u3•v3∗x3•u2** **∗u3∗u2** +(**u1•v3∗x1•u3** **+u2•v3∗x2•u3** **+u3•v3∗x3•u3** **∗u3∗u3**
Division	
A1•A2^{-1}	(**u1•v1∗(x2∧x3)•u1** **+u2•v1∗(x3∧x1)•u1** **+u3•v1∗(x1∧x2)•u1**) **∗u1∗u1**/det[**A2**] +(**u1•v1∗(x2∧x3)•u2** **+u2•v1∗(x3∧x1)•u2** **+u3•v1∗(x1∧x2)•u2**) **∗u1∗u2**/det[**A2**] +(**u1•v1∗(x2∧x3)•u3** **+u2•v1∗(x3∧x1)•u3** **+u3•v1∗(x1∧x2)•u3**) **∗u1∗u3**/det[**A2**] +(**u1•v2∗(x2∧x3)•u1** **+u2•v2∗(x3∧x1)•u1** **+u3•v2∗(x1∧x2)•u1**) **∗u2∗u1**/det[**A2**] +(**u1•v2∗(x2∧x3)•u2** **+u2•v2∗(x3∧x1)•u2** **+u3•v2∗(x1∧x2)•u2**) **∗u2∗u2**/det[**A2**] +(**u1•v2∗(x2∧x3)•u3**



+u2•v2∗(x3∧x1)•u3
+u3•v2∗(x1∧x2)•u3)
∗u2∗u3/det[**A2**]
+(**u1•v3∗(x2∧x3)•u1**
+u2•v3∗(x3∧x1)•u1
+u3•v3∗(x1∧x2)•u1)
∗u3∗u1/det[**A2**]
+(**u1•v3∗(x2∧x3)•u2**
+u2•v3∗(x3∧x1)•u2
+u3•v3∗(x1∧x2)•u2)
∗u3∗u2/det[**A2**]
+(**u1•v3∗(x2∧x3)•u3**
+u2•v3∗(x3∧x1)•u3
+u3•v3∗(x1∧x2)•u3)
∗u3∗u3/det[**A2**]

Einstein: "You have added many things there, Breton. For Instance you have defined a cross multiplication.

Breton: "And also a outer product multiplication. The cross multiplication is just a restatement of our earlier definition when we first discussed the origin. The outer product multiplication comes from our earlier discussion of matrix multiplication.

Solution of vector equations

Einstein: "So can we also solve vector equations involving these multiplications? For instance, if

x∧v1 = v2

with **v1** and **v2** known, what is **x**?

Breton: "May I try something a little easier. How about

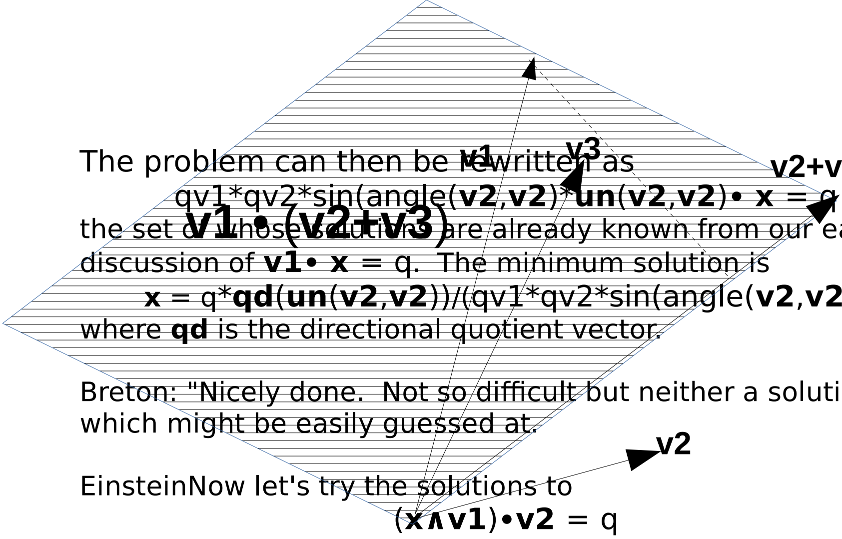
(**v1∧v2**)• **x** = q

where **v1** and **v2** are given vectors and q a given scalar.

Newton: "That's not so difficult. The equation is a triple product. The cross product **v1∧v2** equals

qv1*qv2*sin(angle(**v2**,**v2**)*****un**(**v2**,**v2**)

where **un** is a unit vector orthogonal to both **v1** and **v2.**



The problem can then be rewritten as

qv1*qv2*sin(angle(**v2**,**v2**)*un(**v2**,**v2**)• **x** = q

the set of whose solutions are already known from our earlier discussion of **v1**• **x** = q. The minimum solution is

x = q*qd(**un**(**v2**,**v2**))/(qv1*qv2*sin(angle(**v2**,**v2**))

where **qd** is the directional quotient vector.

Breton: "Nicely done. Not so difficult but neither a solution which might be easily guessed at.

EinsteinNow let's try the solutions to

(**x∧v1**)•**v2** = q

Newton: "Much more difficult. Breton, have you any suggestions.

Breton: "I'd like to introduce another matrix definition which could open a different path to the solution. Remember xx suggested that cross multiplication could be expressed with a matrix. Let us now define that matrix.

Definition (curl vector operator)

Given

v = v∗(c1∗**u1**+c2∗**u2**+c3∗**u3**)

then

C(**v**) ≡ v∗(−c3∗**u1**∗**u2** + c2∗**u1**∗**u2**
+c3∗**u2**∗**u1** − c2∗**u2**∗**u3**
−c2∗**u3**∗**u1** + c1∗**u3**∗**u2**)

is called the **curl vector operator**.

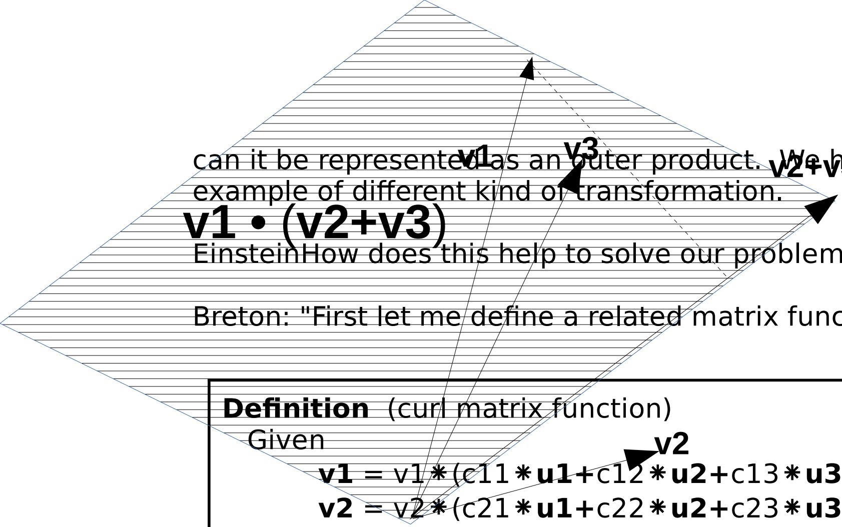
end of definition

EinsteinWhy do you call it an operator?

Breton: "The curl vector operator can be written as a matrix

$$\mathbf{C}(\mathrm{v}) = \mathrm{v} * \begin{bmatrix} 0 & -\mathrm{c}3 & \mathrm{c}2 \\ \mathrm{c}3 & 0 & -\mathrm{c}1 \\ -\mathrm{c}2 & \mathrm{c}1 & 0 \end{bmatrix}$$

The determinant of C(**v**) is zer0 so it has no inverse; neither



can it be represented as an outer product. We have here an example of different kind of transformation.

EinsteinHow does this help to solve our problem?

Breton: "First let me define a related matrix function.

> **Definition** (curl matrix function)
> Given
> $$\mathbf{v1} = v1 * (c11 * \mathbf{u1} + c12 * \mathbf{u2} + c13 * \mathbf{u3})$$
> $$\mathbf{v2} = v2 * (c21 * \mathbf{u1} + c22 * \mathbf{u2} + c23 * \mathbf{u3})$$
>
> then
>
> $$c(\mathbf{v1} * \mathbf{v2}) \equiv \mathbf{v1} \wedge \mathbf{v2}$$
>
> is called the **curl matrix function**.
>
> end of definition

The curl vector operator and the curl matrix function are related as follows:

$$\mathbf{v1} \wedge \mathbf{v2} = c(\mathbf{v1} * \mathbf{v2}) = \mathbf{v1} \bullet C(\mathbf{v2})$$

They will likely find use when dealing with cross products.

EinsteinSo let us see if the help with our problem $(\mathbf{x} \wedge \mathbf{v1}) \bullet \mathbf{v2} = q$!

Breton: "We can rewrite the problem as

$$\mathbf{x} \bullet C(\mathbf{v1}) \bullet \mathbf{v2} = q$$

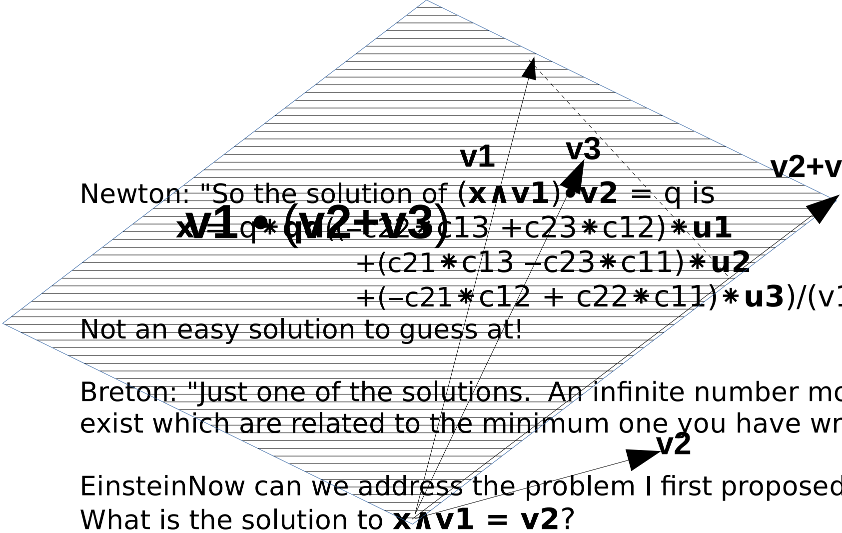
which can be expanded into

$$v1 * v2 * \mathbf{x} \bullet \begin{bmatrix} 0 & -c13 & c12 \\ c13 & 0 & -c11 \\ -c12 & c11 & 0 \end{bmatrix} \bullet \begin{bmatrix} c21 \\ c22 \\ c23 \end{bmatrix} = q$$

Now let's perform the matrix multiplication on the right to obtain

$$\begin{aligned} v1 * v2 * \mathbf{x} \bullet (&(-c22 * c13 + c23 * c12) * \mathbf{u1} \\ &+ (c21 * c13 - c23 * c11) * \mathbf{u2} \\ &+ (-c21 * c12 + c22 * c11) * \mathbf{u3}) \\ &= q \end{aligned}$$

which is just our familiar solution of $\mathbf{v} \bullet \mathbf{x} = q$ in a different garb.



Newton: "So the solution of (**x∧v1**)•**v2** = q is
x = q∗q∗(–c22∗c13 +c23∗c12)∗**u1**
+(c21∗c13 –c23∗c11)∗**u2**
+(–c21∗c12 + c22∗c11)∗**u3**)/(v1∗v2)
Not an easy solution to guess at!

Breton: "Just one of the solutions. An infinite number more exist which are related to the minimum one you have written.

EinsteinNow can we address the problem I first proposed? What is the solution to **x∧v1 = v2**?

Newton: "Let me try. The problem can be restated as

x•C(**v1**) = **v2**

so **x** can be found by simply inverting C(**v1**)!

Breton: "Except that its determinant equals zero and so C(**v1**) has no inverse.

Newton: "Then the equation has no solutions!

Breton: "If **v2** = **0** then **x** = **0** is a solution.

Newton: "True enough. So are there other solutions?

Breton: "Let me try.

x∧v1 = v2

x∗**ux∧uv1 =** v2∗**uv2 /**v1

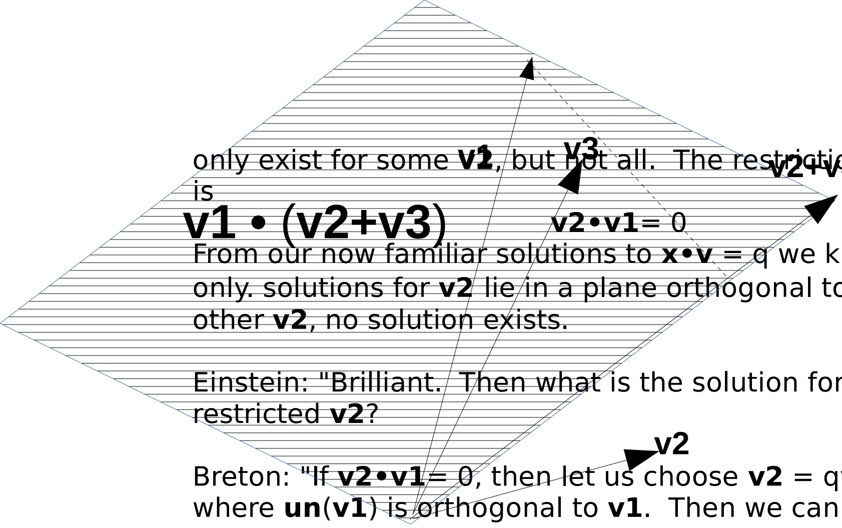
Some solutions are readily apparent. If **v2** = **0**, but not **v1**, then **x** = **0** is the only solution. If **v1** = **0**, but not **v2**, then no solution for **x** exists. If both **v2** = **0** and **v1** = **0**, then **x** may be any vector at all.

EinsteinWhere does that leave us?

Breton: "With the knowledge that solutions not only depend on the directions of **v1** and **v2**, but also their magnitudes.

Einstein: "Sometimes it's easy, any vector will do; other times it's impossible, no vector will do.

Breton: "Try thinking about it this way. Suppose **x** is a solution. Then **v2** must be orthogonal to both **v1** and **x**. So solutions



only exist for some **v1**, but not all. The restriction on **v2**, then is

v2•v1= 0

From our now familiar solutions to **x•v** = q we know that the only. solutions for **v2** lie in a plane orthogonal to **v1**. For any other **v2**, no solution exists.

Einstein: "Brilliant. Then what is the solution for a given restricted **v2**?

Breton: "If **v2•v1**= 0, then let us choose **v2** = qv2∗**un**(**v1**) where **un**(**v1**) is orthogonal to **v1**. Then we can rewrite our equation as

x∧v1 = qv2∗**un**(**v1**)

Now we see that **x** has to be orthogonal to **un**(**v1**) as well. So then both

x∧v1 =qx qv1∗sin(angle(**x**,**v1**))∗**un**(**v1**)
= qv2∗**un**(**x**,**v1**)

define a solution.

Newton: "So

qx **=** q2/(qv1∗sin(angle(**x**,**v1**)))

and

un(**v1**)**= un**(**x**,**v1**)

Einstein: "Which still does not define qx since it depends on angle(**x**,**v1**)!

Breton: "What it does define is the entire set of solutions. Any solution for **x** must satisfy simultaneously

x∧v1 =qx∗qv1∗sin(angle(**x**,**v1**))∗**un**(**x**,**v1**)

and **x∧v1 =** qv2∗**uv2**

All solutions lie in a plane orthogonal to **uv2**, but not all such vectors are solutions, but only those who satisfy

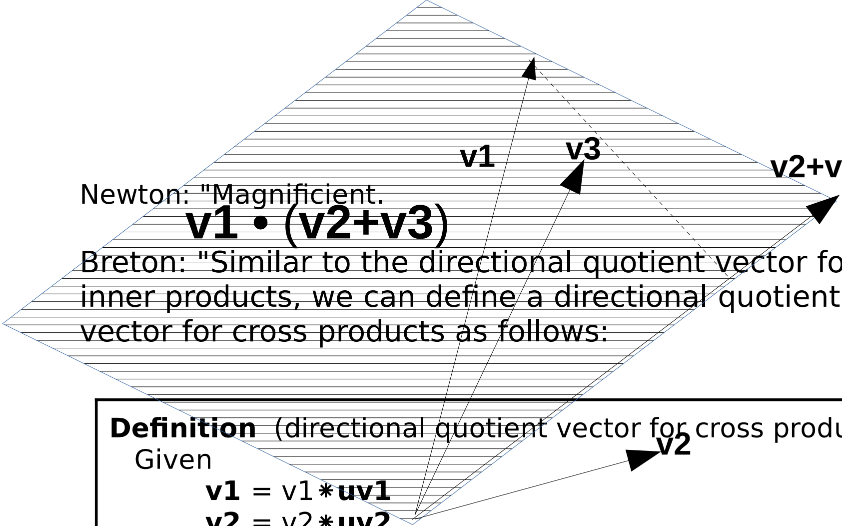
qx∗sin(angle(**x**,**v1**)) = qv2/ qv1

A curve in the plane orthogonal to **v2** designates the entire set of solutions. The whole set of solutions is thus

{**x|x** = (qv2/(qv1∗sin(angle(**x**,**v1**))))∗**un**(**v1,v2**)}

Among these solutions there is one which minimizes qx, namely the one that maximizes sin(angle(**x**,**v1**)), specifically the one for which sin(angle(**x**,**v1**)) = 1. For this minimum solution

x = (qv2/qv1)∗**un**(**v1,v2**)



Newton: "Magnificient.

Breton: "Similar to the directional quotient vector for inner products, we can define a directional quotient vector for cross products as follows:

Definition (directional quotient vector for cross products)
Given
v1 = v1∗**uv1**
v2 = v2∗**uv2**
un(**v1**,**v2**) = **uv1∧uv2**
then
qd(**v1**,**v2**) ≡ v2∗**un**(**v1**,**v2**)/v2
end of definition

Newton: "Then

qd(**u1**,**u2**) = **u3**
qd(u2,u3) = **u1**
qd(u3,u1) = **u2**

Breton: "Exactly**.**

Einstein: "How about outer products?

Breton: "The problem can formulated in any of three ways.

x•v1∗v2=**v3**
v1•x∗v2=**v3**
v1•v2∗x=**v3**

Newton: "The first two formulations are identical since **x•v1**=**v1•x**.

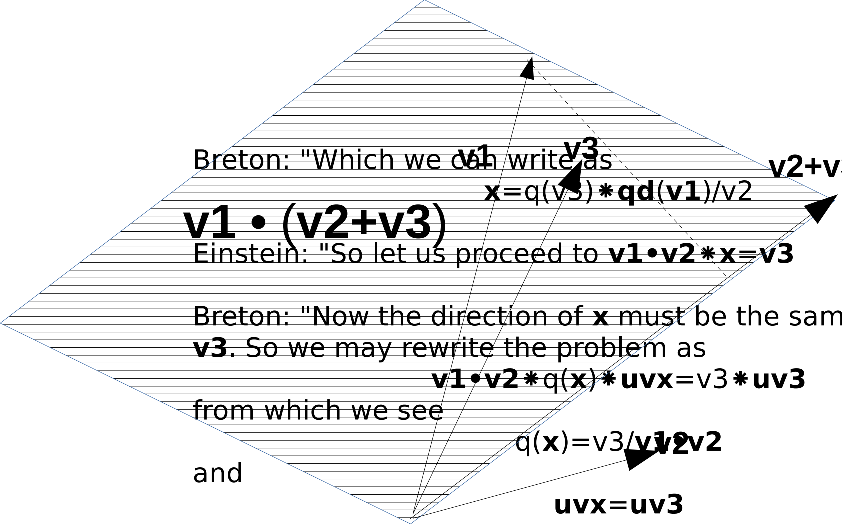
Breton: "I stand corrected. There are only two possible formulations. Let us start with the first one.

x•v1∗v2=q(v3)∗**v2 = v3**

so **v3** must have the same direction as **v2**. Furthermore

x•v1∗v2=q(v3)

Newton: "Whose solutions for **x** we already know, including a minimum one.



Breton: "Which we can write as

x=q(v3)∗**qd**(**v1**)/v2

Einstein: "So let us proceed to **v1•v2∗x=v3**

Breton: "Now the direction of **x** must be the same as that of **v3**. So we may rewrite the problem as

v1•v2∗q(**x**)∗**uvx**=v3∗**uv3**

from which we see

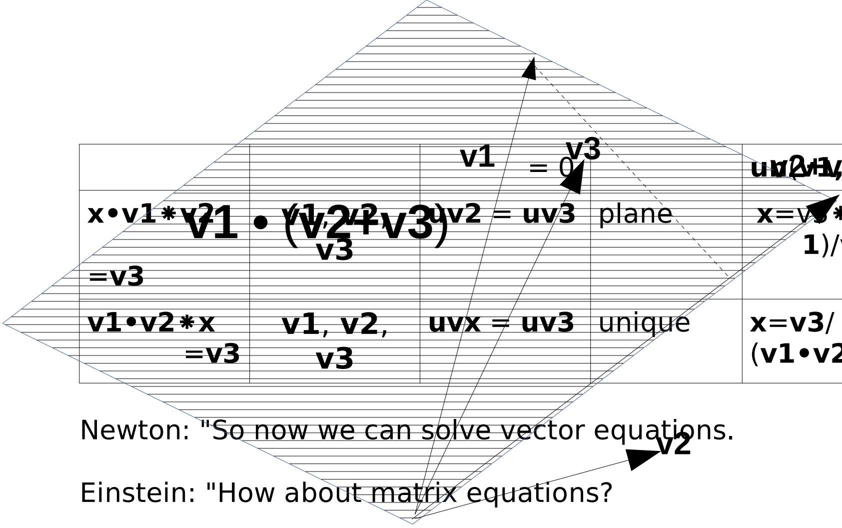
q(**x**)=v3/**v1∗v2**

and

uvx=**uv3**

Einstein: "So for this problem there is only one unique answer. For others, many solutions exist. Newton would you create a table showing these differences.

Equation	given	restrictions	solutions	minimum
x•v1 = q1	**v1**,q1	none	plane	**x** = q1∗**qd**(**v1**)
(**v1∧v2**) • **x** = q	**v1**,**v2**,q	none	plane	**x** = q * **qd**(**un** (**v1**,**v2**)) /(qv1*qv2 *sin(angle (**v2**,**v2**))
(**x∧v1**) •**v2** = q	**v1**,**v2**,q	**v2•v1**= 0	plane	**x** = q∗**qd** ((c23∗c12 −c22∗c13) ∗**u1** +(c21∗c13 −c23∗c11) ∗**u2** +(c22∗c11 −c21∗c12) ∗**u3**) /(v1∗v2)
x∧v1 = **v2**	**v1**, **v2**	**v2•v1**=0; **x•un**(**v1**)	curve in plane	**x** = (qv2/qv1)∗



		= 0		**uv**(**v1**,**v2**)
x•v1∗v2 =**v3**	**v1**, **v2**, **v3**	**uv2** = **uv3**	plane	**x**=**v3∗qd**(**v1**)/v2
v1•v2∗x =**v3**	**v1**, **v2**, **v3**	**uvx** = **uv3**	unique	**x**=**v3**/(**v1•v2**)

Newton: "So now we can solve vector equations.

Einstein: "How about matrix equations?

Breton: "Since we have formed an algebra of matrices, we should be able to solve matrix equations too.

Einstein: "Sounds like a promise. Deliver!

Breton: "Let's start with some matrix, **A**, a given vector **v**, and a matrix equation

$$\mathbf{x} \bullet \mathbf{A} = \mathbf{v}$$

and ask for the unknown vector **x**.

Newton: "That's easy. Simply find the inverse of the matrix. Then

$$\mathbf{x} = \mathbf{v} \bullet \mathbf{A}^{-1}$$

Breton: "That will do for matrices with inverses. What about those without inverses, those whose determinants equal zero.

Einstein: "We've seen just a case with the outer product. If

$$\mathbf{A} = \mathbf{v1} \ast \mathbf{v2}$$

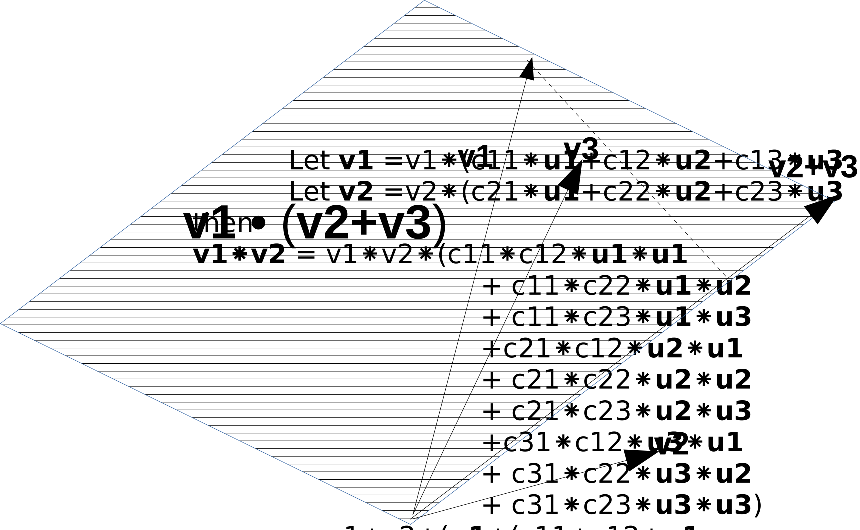
an outer product, it has no inverse, but the solution to **x•v1∗v2** is any vector parallel to **v2**.

Newton: "Show me that outer products have not inverse.

Einstein: "If they did, then the solution would be the unique solution you just demonstrated.

Newton: "Show me that the determinant of any outer product equals zero.

Breton: "All right, let me do it. Please pay close attention to the manipulations.



Let **v1** =v1∗(c11∗**u1**+c12∗**u2**+c13∗**u3**
Let **v2** =v2∗(c21∗**u1**+c22∗**u2**+c23∗**u3**
Then
v1∗v2 = v1∗v2∗(c11∗c12∗**u1∗u1**
+ c11∗c22∗**u1∗u2**
+ c11∗c23∗**u1∗u3**
+c21∗c12∗**u2∗u1**
+ c21∗c22∗**u2∗u2**
+ c21∗c23∗**u2∗u3**
+c31∗c12∗**u3∗u1**
+ c31∗c22∗**u3∗u2**
+ c31∗c23∗**u3∗u3**)
= v1∗v2∗(**u1**∗(c11∗c12∗**u1**
+ c11∗c22∗**u2**
+ c11∗c23∗**u3**)

+**u2**∗(c21∗c12∗**u1**
+ c21∗c22∗**u2∗u2**
+ c21∗c23∗**u2∗u3**)
+**u3**∗(c31∗c12∗**u1**
+ c31∗c22∗**u2**
+c31∗c23∗**u3**))

Therefore
det[**v1∗v2**]=(c11∗c12∗**u1**+ c11∗c22∗**u2**+ c11∗c23∗**u3**)
^(c21∗c12∗**u1**+ c21∗c22∗**u2**+ c21∗c23∗**u3**)
•(c31∗c12∗**u1**+ c31∗c22∗**u2**+ c31∗c23∗**u3**)
= (c11∗c12∗ c21∗c22∗**u3** − c11∗c12∗ c31∗c23∗**u2**
−c11∗c22∗ c21∗c12∗**u3** + c11∗c22∗ c21∗c23∗**u1**
+c11∗c23∗ c31∗c12∗**u2** − c11∗c23∗ c21∗c22∗**u1)**
•(c31∗c12∗**u1**+ c31∗c22∗**u2**+ c31∗c23∗**u3**)
= 0
So any outer product has a determinant equal to zero.

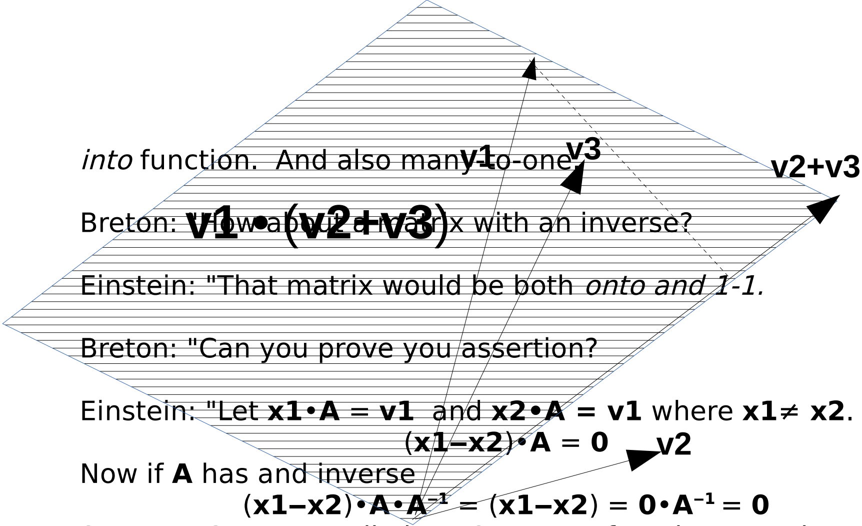
Einstein: "Thank you. Perhaps you can deliver on your promise after all.

Breton: "The promise recognizes distinctions in the set of matrices. A matrix can be seen as as function

A:**V3** ⟶ **V3**

so we can ask functional questions about it. Is an outer product injective or surjective?

Einstein: "I like the terminology *into* or *onto.* Since the outer product maps any vector into a given direction, it must be an



into function. And also many-to-one.

Breton: "How about a matrix with an inverse?

Einstein: "That matrix would be both *onto and 1-1.*

Breton: "Can you prove you assertion?

Einstein: "Let $\mathbf{x1}\bullet\mathbf{A} = \mathbf{v1}$ and $\mathbf{x2}\bullet\mathbf{A} = \mathbf{v1}$ where $\mathbf{x1} \neq \mathbf{x2}$. Then

$$(\mathbf{x1}-\mathbf{x2})\bullet\mathbf{A} = \mathbf{0}$$

Now if **A** has and inverse

$$(\mathbf{x1}-\mathbf{x2})\bullet\mathbf{A}\bullet\mathbf{A}^{-1} = (\mathbf{x1}-\mathbf{x2}) = \mathbf{0}\bullet\mathbf{A}^{-1} = \mathbf{0}$$

So $\mathbf{x1} = \mathbf{x2}$, a contradiction. So **A** as a function must be 1-1.
Next let **v1** be any vector. Then

$$\mathbf{v1}\bullet\mathbf{A}^{-1} = \mathbf{x1}$$

for some vector **x1**. For **x1** then

$$\mathbf{x1}\bullet\mathbf{A} = \mathbf{v1}$$

and so **A** as a function must be onto.

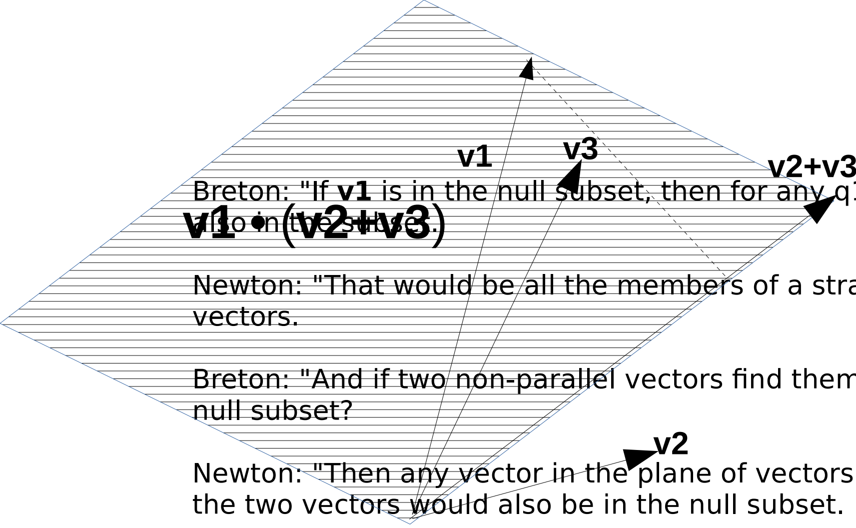
Breton: "Well proven. Matrices as functions then may be either 1-1 and onto, or otherwise, that is, not 1-1 not onto. If **A** = [**0**] for instance, it would map any vector of the domain into the **0** vector of the range. This is an example of a matrix as a constant function.

Newton: "How can we distinguish between the many types of matrices which are not 1-1 and onto?

To examine further the categories of mat rices let me offer the following definition.

> **Definition** (null set of a matrix)
> Given
> **A**, a matrix
> then
>
> $$\mathbf{N}(\mathbf{A}) \equiv \{\mathbf{v}|\mathbf{v} \text{ is a vector such that } \mathbf{v}\bullet\mathbf{A} = \mathbf{0}\}$$
>
> is called the **null subset** of **A**
> end of definition

Einstein: "Then the vector **0** is a member of null subset of any matrix.



Breton: "If **v1** is in the null subset, then for any q1, q1∗**v1** is also in the subset.

Newton: "That would be all the members of a straight line of vectors.

Breton: "And if two non-parallel vectors find themselves in the null subset?

Newton: "Then any vector in the plane of vectors containing the two vectors would also be in the null subset.

Breton: "What is the null subset of a matrix with an inverse?

Newton: "Only the vector **0**.

Breton: "So any given matrix can be categorized as one whose null subset is either a vector line, plane, the whole set of vectors, or simply the vector **0**. We label these subsets with a function called dimension, whose value are

subset **0**	dimension = 0
a line	dimension = 1
a plane	dimension = 2
V3	dimension = 3

The entire set of matrices are divided into four subsets each characterized by a dimension.

Newton: "Whereas the partitions of quotient numbers had only two characterizations—0 and line.

Breton: "Yes, you see the similarities. Do you remember the definition of restricted subsets? Note that each of these partitions,-- lines, planes, or v3 entire-- comes with its restricted algebra.

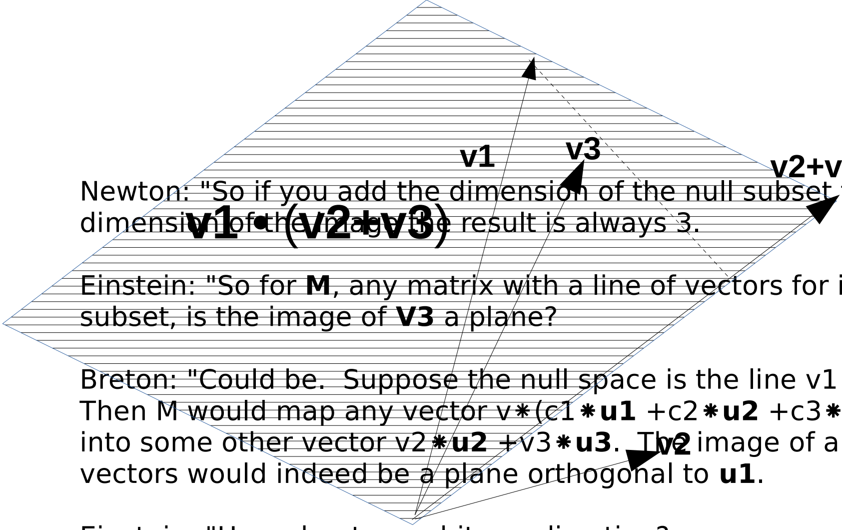
E,And the 0 in quotient numbers evolves into the [**0**] of matrices. How do the subsets relate to **V3**?

Breton: "We already know some answers. For M, any matrix with a **0** null subset,

$$\mathbf{M}: \mathbf{V3} \longrightarrow \mathbf{V3}$$

For M, any matrix with **V3** as its null subset,

$$\mathbf{M}: \mathbf{V3} \longrightarrow \mathbf{0}$$



Newton: "So if you add the dimension of the null subset to the dimension of the image the result is always 3.

Einstein: "So for **M**, any matrix with a line of vectors for its null subset, is the image of **V3** a plane?

Breton: "Could be. Suppose the null space is the line v1∗**u1**. Then M would map any vector v∗(c1∗**u1** +c2∗**u2** +c3∗**u3**) into some other vector v2∗**u2** +v3∗**u3**. The image of all such vectors would indeed be a plane orthogonal to **u1**.

Einstein: "How about an arbitrary direction?

Newton: "Then we might as well chosen the origin to have **u1** as the arbitrary direction with an identical result.

Breton: "So again the dimension of the image added to the dimension of the null set equals 3.

Newton: "A similar argument shows the image of a matrix with a plane for a null set would have an image of a line of vectors.

Breton: "We give a name **rank** to the image of **A**. Then we can write for any matrix **A**

dimension(**N**(**A**)) + rank(**A**) = 3

Einstein: "Why not simply say

dimension(**N**(**A**)) + dimension(image(**A**)) = 3?

Breton: "Acceptable, of course. It's just a bit more convenient to talk about the solutions of matrix equations for a matrix of rank 2, than one whose image has dimension of 2.

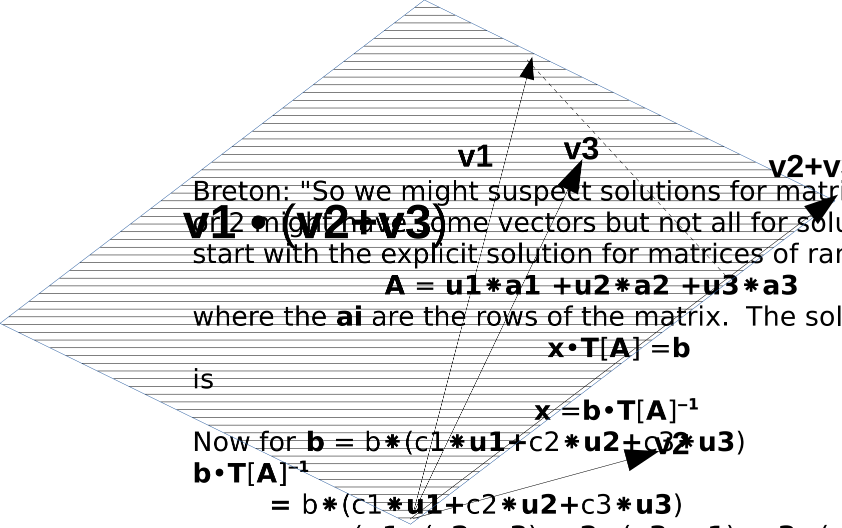
Einstein: "Proceed then to the solutions of matrix equations.

Breton: "Let's start with the equation

x•**A** =**b**

where the matrix **A** and the vector **b** are given. We seek a solution for **x**.

Newton: "We already have solutions for matrices of rank 0 or rank3. For a matrix of rank 3 only one vector is a solution; for a matrix of rank 0 any vector is a solution.



Breton: "So we might suspect solutions for matrices of rank 1 or 2 might have some vectors but not all for solutions. Let's start with the explicit solution for matrices of rank3. Let

A = **u1**⁕**a1** +**u2**⁕**a2** +**u3**⁕**a3**

where the **ai** are the rows of the matrix. The solution for

x•**T**[**A**] =**b**

is

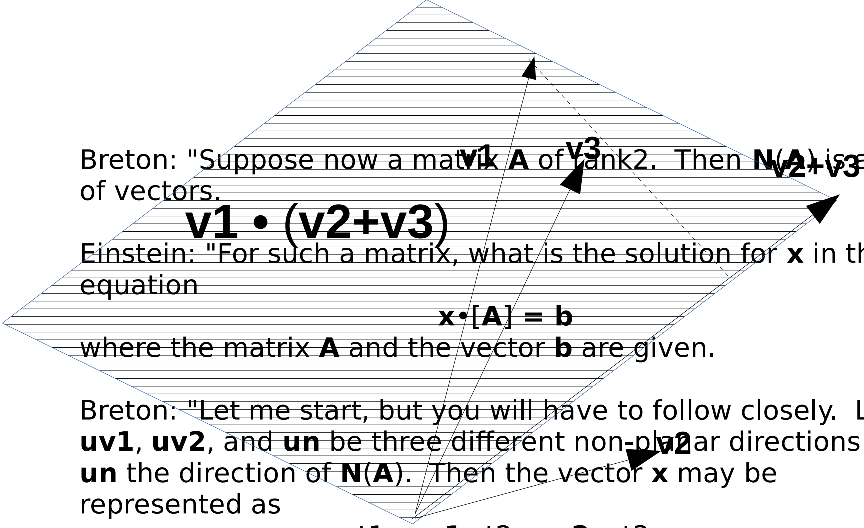
x =**b**•**T**[**A**]$^{-1}$

Now for **b** = b⁕(c1⁕**u1**+c2⁕**u2**+c3⁕**u3**)
b•**T**[**A**]$^{-1}$
  = b⁕(c1⁕**u1**+c2⁕**u2**+c3⁕**u3**)
    •(**u1**⁕(**a2**∧**a3**)+**u2**⁕(**a3**∧**a1**)+**u3**⁕(**a1**∧**a2**))/det[**A**]
  = b⁕((c1⁕(**a2**∧**a3**)+c2⁕(**a3**∧**a1**)+c3⁕(**a1**∧**a2**))/det[**A**]

Einstein: "You've changed the problem by substituting the transpose. Why?

Breton: "The original problem calls for multiplying the unknown vector by the columns of the matrix. By substituting the transpose the solution can be stated in terms of the rows of **A**. If the matrix is given, then so too is its transpose.

Einstein: "Let's move on then to matrices of other ranks.



Breton: "Suppose now a matrix **A** of rank2. Then **N**(**A**) is a line of vectors.

Einstein: "For such a matrix, what is the solution for **x** in the equation

x•[**A**] = **b**

where the matrix **A** and the vector **b** are given.

Breton: "Let me start, but you will have to follow closely. Let **uv1**, **uv2**, and **un** be three different non-planar directions with **un** the direction of **N**(**A**). Then the vector **x** may be represented as

x = t1∗**uv1**+t2∗**uv2** +t3∗**un**

for some t1, t2, and t3. So the solution for

x•[**A**] = (t1∗**uv1**+t2∗**uv2** +t3∗**un**)•**A** = **b**

devolves into a solution for t1 and t2 since t3∗**un**•[**A**] = **0**

Furthermore **b** must be constrained to lie outside **N**(**A**) since if **b**=t3∗**un** contradicts the assumption that **uv1**•**A** lies outside **N**(**A**). Assuming then.

b = b1∗**u1**+b2∗**u2** +b3∗**u3**

b•[**A**] = (b1∗a11+ b2∗a21+ b3∗a31)∗**u1**
+(b1∗a12+ b2∗a22+ b3∗a32)∗**u2**
+(b1∗a13+ b2∗a23+ b3∗a33)∗**u3**
≠ **0**

If **b**= **0** then **x = 0** is the only solution.

Newton: "What is image of **A**?

Breton: "The image is all vectors except those in the null set of **A,** that is, **A − N**(**A**), except that **0** belongs to both the image and the null set.

We might also note that if **x1** and **x2** are solutions, then **x1−x2** lies in the null set of **A** since

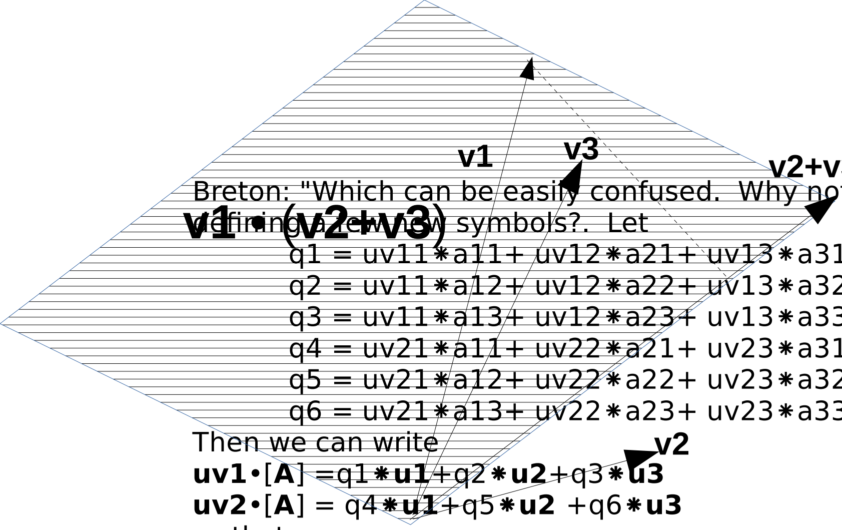
(x1−x2)•A=x1•A −x2•A = **b−b = 0**.

We can also also calculate some components of the equation as

uv1•[**A**] = (uv11∗a11+ uv12∗a21+ uv13∗a31)∗**u1**
+(uv11∗a12+ uv12∗a22+ uv13∗a32)∗**u2**
+(uv11∗a13+ uv12∗a23+ uv13∗a33)∗**u3**

uv2•[**A**] = (uv21∗a11+ uv22∗a21+ uv23∗a31)∗**u1**
+(uv21∗a12+ uv22∗a22+ uv23∗a32)∗**u2**
+(uv21∗a13+ uv22∗a23+ uv23∗a33)∗**u3**

Newton: "That's a porridge of symbols.



Breton: "Which can be easily confused. Why not simplify by defining a few new symbols?. Let

q1 = uv11∗a11+ uv12∗a21+ uv13∗a31
q2 = uv11∗a12+ uv12∗a22+ uv13∗a32
q3 = uv11∗a13+ uv12∗a23+ uv13∗a33
q4 = uv21∗a11+ uv22∗a21+ uv23∗a31
q5 = uv21∗a12+ uv22∗a22+ uv23∗a32
q6 = uv21∗a13+ uv22∗a23+ uv23∗a33

Then we can write
uv1•[**A**] =q1∗**u1**+q2∗**u2**+q3∗**u3**
uv2•[**A**] = q4∗**u1**+q5∗**u2** +q6∗**u3**
so that
(t1∗**uv1+**t2∗**uv2 +**t3∗**un**)•**A**
=t1∗**uv1•A+**t2∗**uv2•A +0**
=t1∗**(**q1∗**u1**+q2∗**u2**+q3∗**u3**)
+ t2∗(q4∗**u1**+q5∗**u2** +q6∗**u3**)
= b1∗**u1+**b2∗**u2 +**b3∗**u3**

Thus,

t1∗q1 + t2∗q4= b1
t1∗q2 + t2∗q5= b2
t1∗q3 + t2∗q6= b3

So we have three different equations which can be solved for two unknowns. We can rewrite the equations as

t2 = (b1 − t1∗q1)/q4
t2 = (b2 − t1∗q2)/q5
t2 = (b3 − t1∗q3)/q6

First let's solve for t1 from the first two equations.

(b1 − t1∗q1)/q4 = (b2 − t1∗q2)/q5

so that

(b1 − t2∗q4)/q1 = (b2 − t2∗q5)/q2

and so

t2∗q5)/q2 − t2∗q4)/q1= (b2/q2−b1/q1

and so

t2∗q5∗q1 − t2∗q4∗q2= (b2∗q1−b1∗q2

and so finally

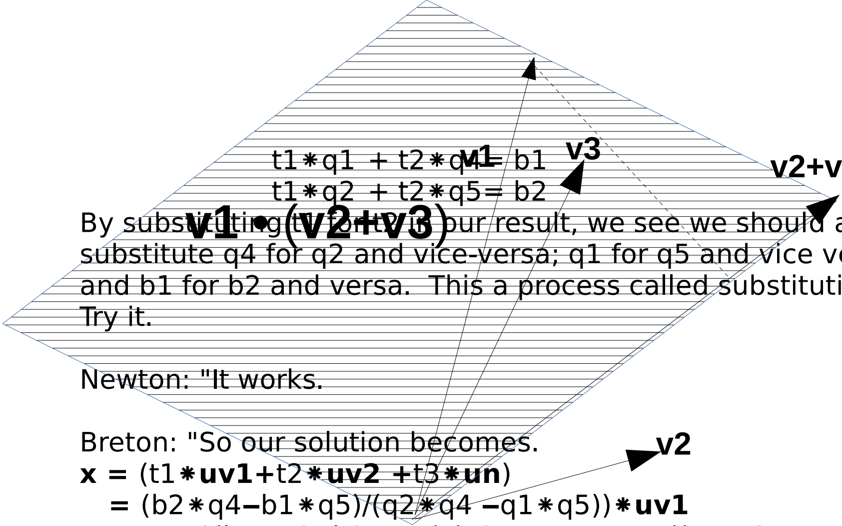
t2 = (b2∗q1−b1∗q2)/(q5∗q1 −q4∗q2)

Similarly,

t1=(b1∗q2−b2∗q1)/(q2∗q4 −q1∗q5)

Newton: "How can you put down the result for t1 so quickly and easily?

Breton: "We are using only the first two equations.



t1∗q1 + t2∗q4= b1
t1∗q2 + t2∗q5= b2

By substituting t1 for t2 in our result, we see we should also substitute q4 for q2 and vice-versa; q1 for q5 and vice versa; and b1 for b2 and versa. This a process called substitution. Try it.

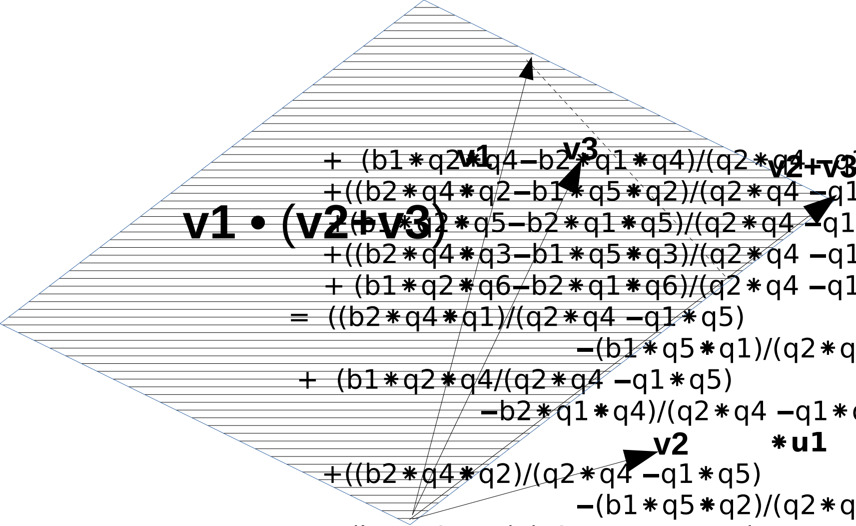
Newton: "It works.

Breton: "So our solution becomes.
x = (t1∗**uv1+**t2∗**uv2 +**t3∗**un**)
= (b2∗q4**−**b1∗q5)/(q2∗q4 **−**q1∗q5))∗**uv1**
+*(b1∗q2**−**b2∗q1)/(q2∗q4 **−**q1∗q5))∗**uv2**
+t3∗**un**)

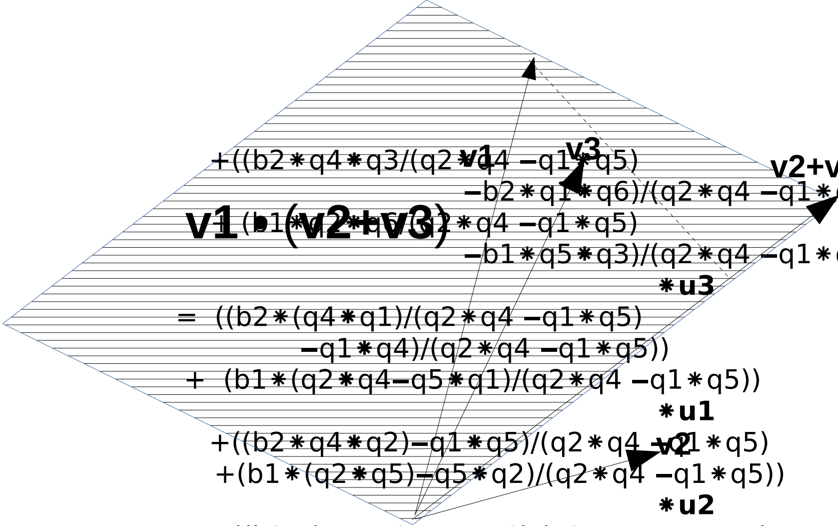
Newton: "Not something we could guess at easily.

Einstein: "Show directly that **x** is a solution.

Breton: "All right. Let us check against the original
x•[**A**] = (t1∗**uv1+**t2∗**uv2 +**t3∗**un**)•[**A**]. Starting with t3 = 0 let us and first calculate some of the components. Remember
uv1•[**A**] =q1∗**u1**+q2∗**u2**+q3∗**u3**
uv2•[**A**] = q4∗**u1**+q5∗**u2** +q6∗**u3**
so
t1∗**uv1•A** = (b2∗q4**−**b1∗q5)∗(q1∗**u1**+q2∗**u2**+q3∗**u3**)
/(q2∗q4 **−**q1∗q5)
= (b2∗q4∗q1**−**b1∗q5∗q1)∗**u1**/(q2∗q4 **−**q1∗q5)
+(b2∗q4∗q2**−**b1∗q5∗q2)∗**u2**/(q2∗q4 **−**q1∗q5)
+(b2∗q4∗q3**−**b1∗q5∗q3)∗**u3**/(q2∗q4 **−**q1∗q5)
t1∗**uv2•A** = (b1∗q2**−**b2∗q1)∗(q4∗**u1**+q5∗**u2** +q6∗**u3)**
/(q2∗q4 **−**q1∗q5)
= (b1∗q2∗q4**−**b2∗q1∗q4)∗**u1**/(q2∗q4 **−**q1∗q5)
+ (b1∗q2∗q5**−**b2∗q1∗q5)∗**u2**/(q2∗q4 **−**q1∗q5)
+ (b1∗q2∗q6**−**b2∗q1∗q6)∗**u3**/(q2∗q4 **−**q1∗q5)
So
(t1∗**uv1+**t2∗**uv2**)•[**A**]
= (b2∗q4∗q1**−**b1∗q5∗q1)∗**u1**/(q2∗q4 **−**q1∗q5)
+(b1∗q2∗q4**−**b2∗q1∗q4)∗**u1**/(q2∗q4 **−**q1∗q5)
+(b2∗q4∗q2**−**b1∗q5∗q2)∗**u2**/(q2∗q4 **−**q1∗q5)
+(b1∗q2∗q5**−**b2∗q1∗q5)∗**u2**/(q2∗q4 **−**q1∗q5)
+(b2∗q4∗q3**−**b1∗q5∗q3)∗**u3**/(q2∗q4 **−**q1∗q5)
+ (b1∗q2∗q6**−**b2∗q1∗q6)∗**u3**/(q2∗q4 **−**q1∗q5)
= ((b2∗q4∗q1**−**b1∗q5∗q1)/(q2∗q4 **−**q1∗q5)



\+ (b1∗q2∗q4−b2∗q1∗q4)/(q2∗q4 −q1∗q5))∗**u1**
+((b2∗q4∗q2−b1∗q5∗q2)/(q2∗q4 −q1∗q5)
\+ (b1∗q2∗q5−b2∗q1∗q5)/(q2∗q4 −q1∗q5))∗**u2**
+((b2∗q4∗q3−b1∗q5∗q3)/(q2∗q4 −q1∗q5)
\+ (b1∗q2∗q6−b2∗q1∗q6)/(q2∗q4 −q1∗q5))∗**u3**
= ((b2∗q4∗q1)/(q2∗q4 −q1∗q5)
−(b1∗q5∗q1)/(q2∗q4 −q1∗q5))
\+ (b1∗q2∗q4/(q2∗q4 −q1∗q5)
−b2∗q1∗q4)/(q2∗q4 −q1∗q5))
∗**u1**
+((b2∗q4∗q2)/(q2∗q4 −q1∗q5)
−(b1∗q5∗q2)/(q2∗q4 −q1∗q5)
+(b1∗q2∗q5)/(q2∗q4 −q1∗q5)
−(b2∗q1∗q5)/(q2∗q4 −q1∗q5))
∗**u2**
+((b2∗q4∗q3/(q2∗q4 −q1∗q5)
−b1∗q5∗q3)/(q2∗q4 −q1∗q5)
\+ (b1∗q2∗q6/(q2∗q4 −q1∗q5)
−b2∗q1∗q6)/(q2∗q4 −q1∗q5))
∗**u3**
= ((b2∗q4∗q1)/(q2∗q4 −q1∗q5)
−b2∗q1∗q4)/(q2∗q4 −q1∗q5))
\+ (b1∗q2∗q4/(q2∗q4 −q1∗q5)
−(b1∗q5∗q1)/(q2∗q4 −q1∗q5))
∗**u1**
+((b2∗q4∗q2)/(q2∗q4 −q1∗q5)
−(b2∗q1∗q5)/(q2∗q4 −q1∗q5)
+(b1∗q2∗q5)/(q2∗q4 −q1∗q5)
−(b1∗q5∗q2)/(q2∗q4 −q1∗q5))
∗**u2**
+((b2∗q4∗q3/(q2∗q4 −q1∗q5)
−b2∗q1∗q6)/(q2∗q4 −q1∗q5))
\+ (b1∗q2∗q6/(q2∗q4 −q1∗q5)
−b1∗q5∗q3)/(q2∗q4 −q1∗q5)
∗**u3**
= ((b2∗(q4∗q1)/(q2∗q4 −q1∗q5)
−q1∗q4)/(q2∗q4 −q1∗q5))
\+ (b1∗(q2∗q4−q5∗q1)/(q2∗q4 −q1∗q5))
∗**u1**
+((b2∗q4∗q2)/(q2∗q4 −q1∗q5)
−(b2∗q1∗q5)/(q2∗q4 −q1∗q5)
+(b1∗q2∗q5)/(q2∗q4 −q1∗q5)
−(b1∗q5∗q2)/(q2∗q4 −q1∗q5))
∗**u2**



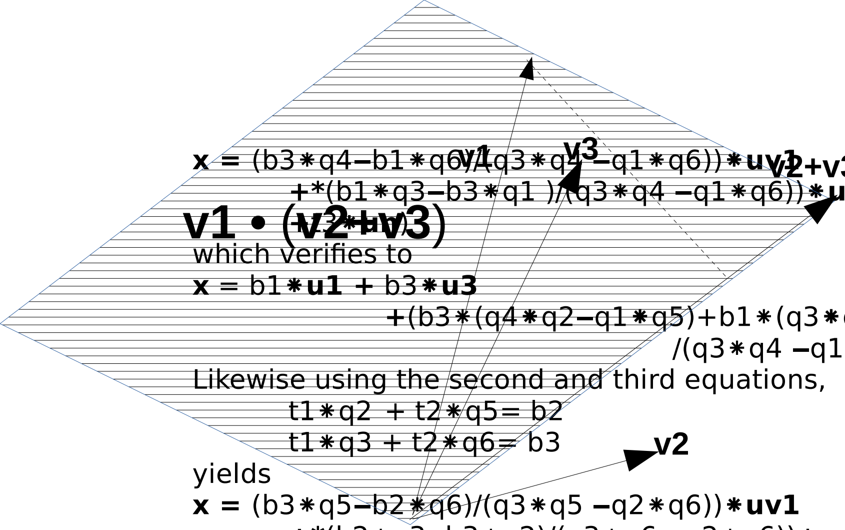
+((b2∗q4∗q3/(q2∗q4 −q1∗q5)
−b2∗q1∗q6)/(q2∗q4 −q1∗q5))
+(b1∗(q2∗q6/(q2∗q4 −q1∗q5)
−b1∗q5∗q3)/(q2∗q4 −q1∗q5)
∗u3

= ((b2∗(q4∗q1)/(q2∗q4 −q1∗q5)
−q1∗q4)/(q2∗q4 −q1∗q5))
+ (b1∗(q2∗q4−q5∗q1)/(q2∗q4 −q1∗q5))
∗u1
+((b2∗q4∗q2)−q1∗q5)/(q2∗q4 −q1∗q5)
+(b1∗(q2∗q5)−q5∗q2)/(q2∗q4 −q1∗q5))
∗u2
+((b2∗(q4∗q3−q1∗q6)/(q2∗q4 −q1∗q5)
+ (b1∗(q2∗q6−q5∗q3)/(q2∗q4 −q1∗q5)
∗u3
= b1∗**u1** + b2∗**u2**
+(b2∗(q4∗q3−q1∗q6)+b1∗(q2∗q6−q5∗q3)
/(q2∗q4 −q1∗q5))∗**u3**

Einstein: "So your **x** is not a solution.

Breton: "It does solve correctly for two of the three components of **b**. Remember we have only used the first two of three equations which do not explicitly reference b3. The complicated scalar for **u3**, may yet evaluate to b3. What do you think the result would be if we had used the first and the third equations?

Newton: "We can try substitution. If
t1∗q1 + t2∗q4= b1
t1∗q2 + t2∗q5= b2
yields
x = (b2∗q4−b1∗q5)/(q2∗q4 −q1∗q5))∗**uv1**
+*(b1∗q2−b2∗q1)/(q2∗q4 −q1∗q5))∗**uv2**
+t3∗**un**)
for a solution
which verifies to
b1∗**u1** + b2∗**u2**
+(b2∗(q4∗q3−q1∗q6)+b1∗(q2∗q6−q5∗q3)
/(q2∗q4 −q1∗q5))∗**u3**
then
t1∗q1 + t2∗q4= b1
t1∗q3 + t2∗q6= b3
yields



x = (b3∗q4−b1∗q6)/(q3∗q4 −q1∗q6))∗**uv1**

+*(b1∗q3−b3∗q1)/(q3∗q4 −q1∗q6))∗**uv2**

+t3∗**un**)

which verifies to

x = b1∗**u1 +** b3∗**u3**

+(b3∗(q4∗q2−q1∗q5)+b1∗(q3∗q5−q6∗q2)

/(q3∗q4 −q1∗q6))∗**u2**

Likewise using the second and third equations,

t1∗q2 + t2∗q5= b2

t1∗q3 + t2∗q6= b3

yields

x = (b3∗q5−b2∗q6)/(q3∗q5 −q2∗q6))∗**uv1**

+*(b2∗q3−b3∗q2)/(q3∗q6 −q2∗q6))∗**uv2**

+t3∗**un**)

which verifies to

x = b2∗**u2 +** b3∗**u3**

+(b3∗(q5∗q1−q2∗q4)+b2∗(q3∗q4−q6∗q1)

/(q3∗q5 −q2∗q6))∗**u1**

Einstein: "Let's do an example.

Breton: "All right. Let

A = **u1∗u1+u1∗u2+u1∗u3**

+3∗**u2∗u1+**2∗**u2∗u2+u2∗u3**

+3∗**u3∗u1+**3∗**u3∗u2+**3∗**u3∗u3**

The null subset of **A** is {t3∗(3∗**u1−u3**)}

Let

b = u1+u2+u3

Then b1 = 1 = b2 = b3.

Let

uv1 = **u1**

uv2 = **u2**

Then

q1 = uv11∗a11+ uv12∗a21+ uv13∗a31= 1

q2 = uv11∗a12+ uv12∗a22+ uv13∗a32= 1

q3 = uv11∗a13+ uv12∗a23+ uv13∗a33= 1

q4 = uv21∗a11+ uv22∗a21+ uv23∗a31= 3

q5 = uv21∗a12+ uv22∗a22+ uv23∗a32= 2

q6 = uv21∗a13+ uv22∗a23+ uv23∗a33 = 1

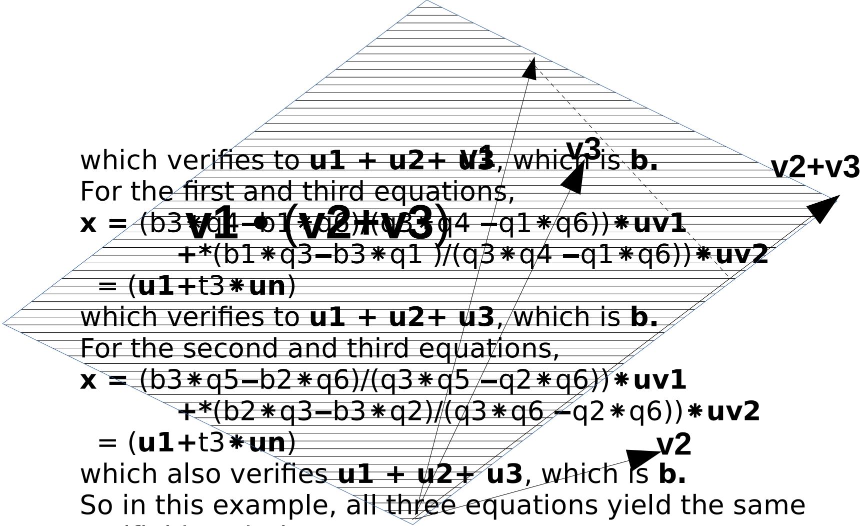
So

x = (b2∗q4−b1∗q5)/(q2∗q4 −q1∗q5))∗**uv1**

+*(b1∗q2−b2∗q1)/(q2∗q4 −q1∗q5))∗**uv2**

+t3∗**un**)

= (**u1+**t3∗**un**)



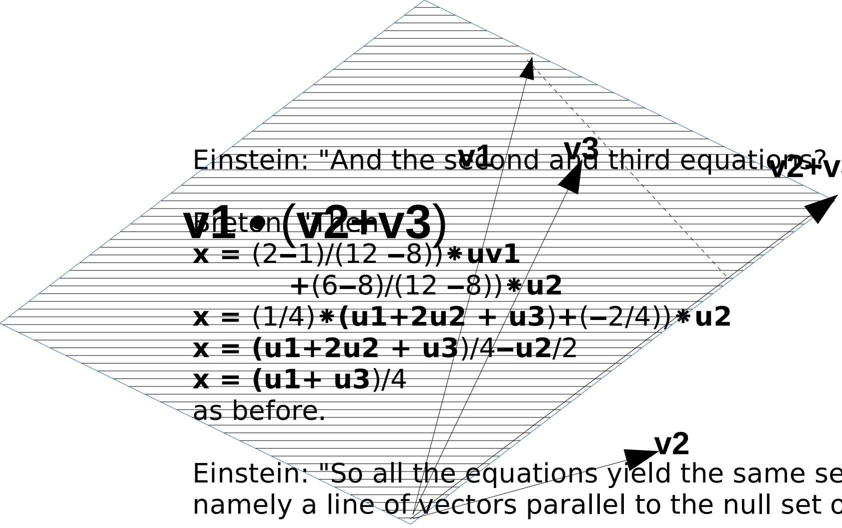
which verifies to **u1 + u2+ u3**, which is **b.**
For the first and third equations,
x = (b3∗q4−b1∗q6)/(q3∗q4 −q1∗q6))∗**uv1**
 +*(b1∗q3−b3∗q1)/(q3∗q4 −q1∗q6))∗**uv2**
 = (**u1+**t3∗**un**)
which verifies to **u1 + u2+ u3**, which is **b.**
For the second and third equations,
x = (b3∗q5−b2∗q6)/(q3∗q5 −q2∗q6))∗**uv1**
 +*(b2∗q3−b3∗q2)/(q3∗q6 −q2∗q6))∗**uv2**
 = (**u1+**t3∗**un**)
which also verifies **u1 + u2+ u3**, which is **b.**
So in this example, all three equations yield the same verifiable solutions.

Einstein: "The example is too simple. Let **uv1** = **u1** +2∗**u2** + **u3**.

Breton: "Then
 q1 = uv11∗a11+ uv12∗a21+ uv13∗a31= 10
 q2 = uv11∗a12+ uv12∗a22+ uv13∗a32= 8
 q3 = uv11∗a13+ uv12∗a23+ uv13∗a33= 6
 q4 = uv21∗a11+ uv22∗a21+ uv23∗a31= 3
 q5 = uv21∗a12+ uv22∗a22+ uv23∗a32= 2
 q6 = uv21∗a13+ uv22∗a23+ uv23∗a33 = 1
Now for the first and second equations,
x = (b2∗q4−b1∗q5)/(q2∗q4 −q1∗q5))∗**uv1**
 +*(b1∗q2−b2∗q1)/(q2∗q4 −q1∗q5))∗**uv2**
 +t3∗**un**)
 = ((**u1+**2∗**u2+u3**)/4) − (**u2**/2)**+**t3∗**un**)
 = ((**u1+u3**)/4)**+**t3∗**un**)
which verifies to **u1 + u2+ u3**, which is **b**.

Einstein: "Try the first and third equations.

Breton: "Then
x = (b3∗q4−b1∗q6)/(q3∗q4 −q1∗q6))∗**uv1**
 +*(b1∗q3−b3∗q1)/(q3∗q4 −q1∗q6))∗**uv2**
x = (3−1)/(18 −10))∗**(u1+2u2 + u3**)
 +*(6−10)/(18 −10))∗**u2**
x = (2/8)∗**(u1+2u2 + u3**)+(−4)/(8)∗**u2**
x = (u1+2u2 + u3)/4−**u2**/2
x = (u1+ u3)/4
as before.



Einstein: "And the second and third equations?

Breton: "Then
x = (2**−**1)/(12 **−**8))**∗uv1**
+(6**−**8)/(12 **−**8))**∗u2**
x = (1/4)**∗(u1+2u2 + u3**)**+**(**−**2/4))**∗u2**
x = (u1+2u2 + u3)/4**−u2**/2
x = (u1+ u3)/4
as before.

Einstein: "So all the equations yield the same set of solutions, namely a line of vectors parallel to the null set of **A**.

Breton: "So the set of solutions can be expressed as
{**x**} = **xm** +{**y|y** is a vector in **N**(**A**)}
where **xm** is orthogonal to **N**(**A**). The solution **xm** would be the one with minimum magnitude. Let us find **xm**.

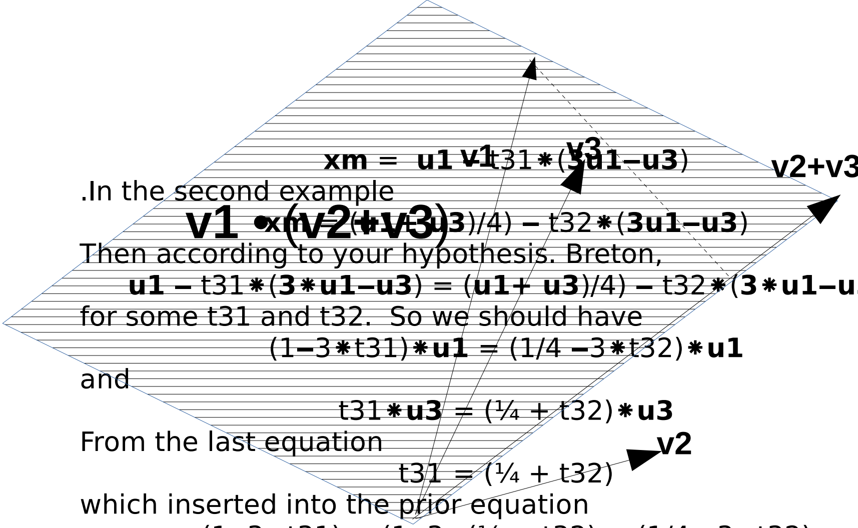
Newton: "That's easy. The vector **y** we have previously expressed as t3**∗un**. Then since any single solution
x1 = **xm** + **y** = **xm** + t3**∗un**
just set t3=0. Then **x1** = **xm**.

Einstein: "Not so. Just look at our examples. In the first simple example
x1 = (**u1+**t3**∗un**)
while in the second example,
x1 = (**u1+ u3**)/4) **+**t3**∗un**
So does **xm** equal **u1** or **u1+ u3**)/4?

Newton: "Breton, what is going on?

Breton: "The examples differ only in the choice of the the choice of **uv1.** The examples show that the the resulting solution depends on the arbitrary choice of **uv1** and **uv2**, suitably restricted. Every solution does fit the mold of
x1 = **xm** + t3**∗un**
for some t3. So for the first example.
u1 = **xm** + t31**∗**(**3u1−u3**)
and for the second example,
(**u1+ u3**)/4) = **xm** + t32**∗**(**3u1−u3**)
should solve for the same **xm**.

Einstein: "In the first example



$$\mathbf{xm} = \mathbf{u1} - t31*(\mathbf{3u1-u3})$$

.In the second example

$$\mathbf{xm} = (\mathbf{u1+u3})/4) - t32*(\mathbf{3u1-u3})$$

Then according to your hypothesis. Breton,

$$\mathbf{u1} - t31*(\mathbf{3*u1-u3}) = (\mathbf{u1+u3})/4) - t32*(\mathbf{3*u1-u3})$$

for some t31 and t32. So we should have

$$(1-3*t31)*\mathbf{u1} = (1/4 - 3*t32)*\mathbf{u1}$$

and

$$t31*\mathbf{u3} = (\tfrac{1}{4} + t32)*\mathbf{u3}$$

From the last equation

$$t31 = (\tfrac{1}{4} + t32)$$

which inserted into the prior equation

$$(1-3*t31) = (1-3*(\tfrac{1}{4} + t32) = (1/4 - 3*t32)$$

which confirms your conjecture.

Breton: "The minimum solution must be orthogonal to **N**(**A**). However,

$$\mathbf{u1}\bullet(\mathbf{3u1-u3}) = 3$$

and

$$((\mathbf{u1+u3})/4)\bullet(\mathbf{3u1-u3}) = (3-1)/4 = 1/2$$

so neither of the the solutions acquired by the examples is the minimum solution.

Newton: "So forget the examples; go to the general case.

Einstein: "If we can't solve for the specific example, we won't be able to solve the more general case.

Breton: "So let's first solve for the example. We know the null set and we know two solutions. For orthogonality we only need two vectors, one from the null set and one from the set of solutions. So let us set the orthogonality equation as

$$(\mathbf{u1+}\ t3*(\mathbf{3*u1-u3}))\bullet(\mathbf{3*u1-u3}) = 0$$

where we take the vector (**u1+** t3∗(**3∗u1−u3**)) for **xm** and the vector **3∗u1−u3** from the null set. We can then solve for t3 and thus calculate **xm.**

Newton: "So then

$(\mathbf{u1+}\ t3*(\mathbf{3*u1-u3}))\bullet(\mathbf{3*u1-u3}) = 0$

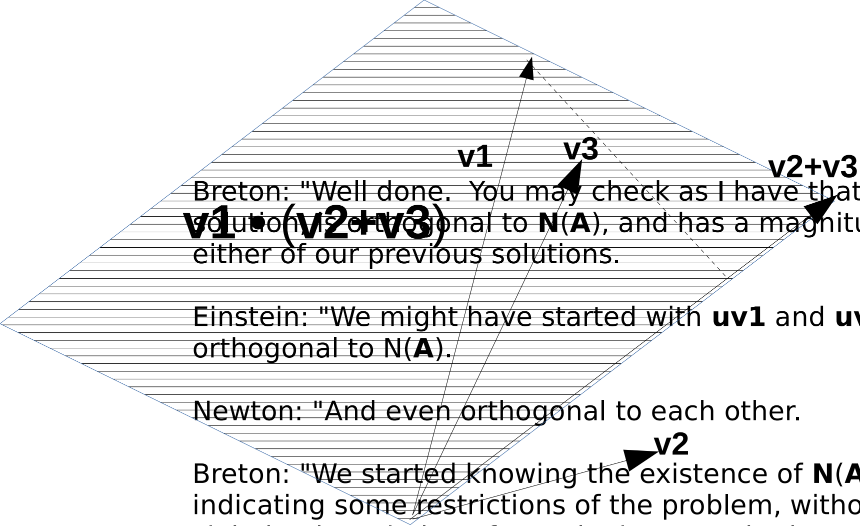
$((3*t3+1)*\mathbf{u1-}\ t3*\mathbf{u3}))\bullet(\mathbf{3*u1-u3}) = 0$

$3*((3*t3+1) + t3)) = 0$

$10*t3 + 3 = 0$

So t3 = −3/10 and

$$\mathbf{xm} = (\mathbf{u1} - 3*(\mathbf{3*u1-u3}))/10 = (\mathbf{u1} + 3*\mathbf{u3})/10$$



Breton: "Well done. You may check as I have that **xm** is a solution, is orthogonal to **N**(**A**), and has a magnitude less than either of our previous solutions.

Einstein: "We might have started with **uv1** and **uv2** orthogonal to N(**A**).

Newton: "And even orthogonal to each other.

Breton: "We started knowing the existence of **N**(**A**), and indicating some restrictions of the problem, without, however, claiming knowledge of any single vector in the null set. That lack of knowledge may have complicated the solution. We could have assumed knowledge of **un**, a knowledge which can be acquired from the matrix. Then **un**•[**A**] **= 0** places a a constraint on the elements of **A**. Furthermore **uv1** and **uv2** can be chosen as you have indicated so that

un•uv1 = 0
un•uv2 = 0
uv1•uv2 = 0

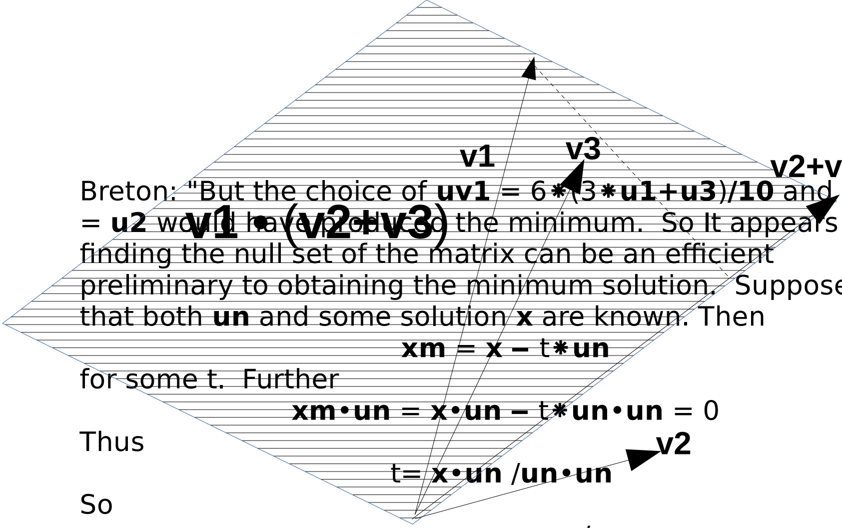
which may have simplified the proof.
In the examples the equation for is given, but the vectors **uv1** and **uv2** are chosen arbitrarily.

Newton: "Let's do the example choosing the inquiring vectors orthogonally. Suppose **uv1** = 3∗**u1+u3** and **uv2** = **u2**. This choice satisfies our conditions.

Einstein: "Then

q1 = uv11∗a11+ uv12∗a21+ uv13∗a31 = 6
q2 = uv11∗a12+ uv12∗a22+ uv13∗a32 = 6
q3 = uv11∗a13+ uv12∗a23+ uv13∗a33 = 6
q4 = uv21∗a11+ uv22∗a21+ uv23∗a31 = 3
q5 = uv21∗a12+ uv22∗a22+ uv23∗a32 = 2
q6 = uv21∗a13+ uv22∗a23+ uv23∗a33 = 1

Then using the first solution with t3 = 0
x = (b2∗q4**−**b1∗q5)/(q2∗q4 **−**q1∗q5))∗**uv1**
+*(b1∗q2**−**b2∗q1)/(q2∗q4 **−**q1∗q5))∗**uv2**
= (3**−**2)/(18 −12))∗**(**3∗**u1+u3**)
+*(6−6)/(18 −12))∗**u2**
= (3∗**u1** +**u3**)/6
So this process while convenient, does not yield the minimum solution.



Breton: "But the choice of **uv1** = 6∗(3∗**u1+u3**)/**10** and **uv2** = **u2** would have produced the minimum. So It appears that finding the null set of the matrix can be an efficient preliminary to obtaining the minimum solution. Suppose then that both **un** and some solution **x** are known. Then

$$\mathbf{xm} = \mathbf{x} - t*\mathbf{un}$$

for some t. Further

$$\mathbf{xm}\bullet\mathbf{un} = \mathbf{x}\bullet\mathbf{un} - t*\mathbf{un}\bullet\mathbf{un} = 0$$

Thus

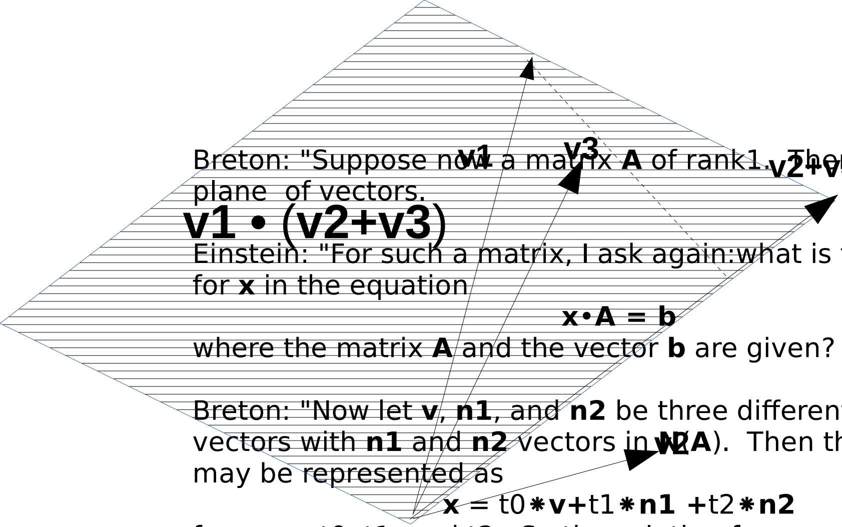
$$t = \mathbf{x}\bullet\mathbf{un}\,/\mathbf{un}\bullet\mathbf{un}$$

So

$$\mathbf{xm} = \mathbf{x} - \mathbf{x}\bullet\mathbf{un}*\mathbf{un}/\mathbf{un}\bullet\mathbf{un}$$
$$= \mathbf{x}\bullet[\mathbf{I} - \mathbf{un}*\mathbf{un}]/\mathbf{un}\bullet\mathbf{un}$$

where **I** is the identity matrix.

Einstein: "How about rank 1 matrices.



Breton: "Suppose now a matrix **A** of rank1. Then **N**(**A**) is a plane of vectors.

Einstein: "For such a matrix, I ask again:what is the solution for **x** in the equation

x•**A = b**

where the matrix **A** and the vector **b** are given?

Breton: "Now let **v**, **n1**, and **n2** be three different non-planar vectors with **n1** and **n2** vectors in **N**(**A**). Then the vector **x** may be represented as

x = t0∗**v+**t1∗**n1 +**t2∗**n2**

for some t0, t1, and t2. So the solution for

x•**A =** (t0∗**v+**t1∗**n1 +**t2∗**n2**)•**A = b**

devolves into a solution for t0 since

(t1∗**n1** +t2∗**n2**•[**A**] **= 0**

Furthermore **b** must be constrained to lie outside **N**(**A**) since otherwise **b** contradicts the assumption that **uv•A** lies outside **N**(**A**). Assuming then.

b = b1∗**u1+**b2∗**u2 +**b3∗**u3**

b•[**A**] = (b1∗a11+ b2∗a21+ b3∗a31)∗**u1**
+(b1∗a12+ b2∗a22+ b3∗a32)∗**u2**
+(b1∗a13+ b2∗a23+ b3∗a33)∗**u3**
≠ **0**

If **b**= **0** then **x = 0** is the only solution.

Newton: "What is image of **A**?

Breton: "The image is all vectors except those in the null set of **A,** that is, **A − N**(**A**), except that **0** belongs to both the image and the null set.

We might also note that if **x1** and **x2** are solutions, then **x1−x2** lies in the null set of **A** since

(x1−x2)•A=x1•A −x2•A = **b−b = 0**.

We can also also calculate

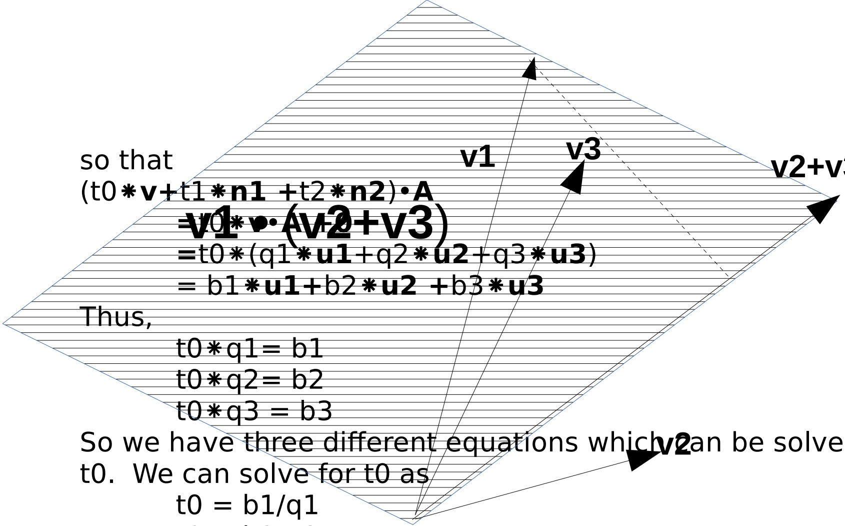
v•[**A**] = (v1∗a11+ v2∗a21+ v3∗a31)∗**u1**
+(v1∗a12+ v2∗a22+ v3∗a32)∗**u2**
+(v1∗a13+ v2∗a23+ v3∗a33)∗**u3**

which we can symbolize by defining a few new symbols as

q1 = v1∗a11+ v2∗a21+ v3∗a31
q2 = v1∗a12+ v2∗a22+ v3∗a32
q3 = v1∗a13+ v2∗a23+ v3∗a33

Then we can write

v•[**A**] =q1∗**u1**+q2∗**u2**+q3∗**u3**



so that
(t0∗**v+**t1∗**n1 +**t2∗**n2**)•**A**
=t0∗**v**•**A**
=t0∗(q1∗**u1**+q2∗**u2**+q3∗**u3**)
= b1∗**u1+**b2∗**u2 +**b3∗**u3**
Thus,
t0∗q1= b1
t0∗q2= b2
t0∗q3 = b3
So we have three different equations which can be solved for t0. We can solve for t0 as
t0 = b1/q1
t0 = b2/q2
t0 = b3/q3
So our solution becomes.
x = (t0∗**v+**t1∗**n1 +**t2∗**n2**)
= (b1/q1)∗**v +**t1∗**n1 +**t2∗**n2**)
or
x = (b2/q2)∗**v +**t1∗**n1 +**t2∗**n2**)
or
x = (b3/q3)∗**v +**t1∗**n1 +**t2∗**n2**)

Einstein: "Now show directly that **x** is a solution.

Breton: "All right. Let us check against the original **x**•**A =** (t0∗**v+**t1∗**n1 +**t2∗**n2**)•**A**. Starting with t1 = 0 and t2 = 0 let us and first calculate
t0∗**v**•**A** =b1∗(q1∗**u1**+q2∗**u2**+q3∗**u3)**/q1
=b1∗**u1**+b1∗q2∗**u2/**q1+b1∗q3∗**u3**/q1

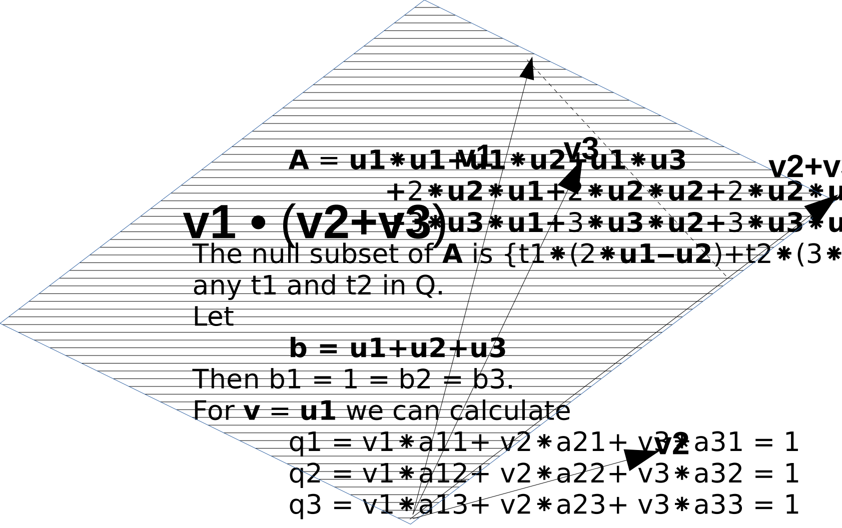
Einstein: "So your **x** is not a solution.

Breton: "But it is! Did you notice,
b1/q1 = b2/q2 = b3/q3
so
b1∗q2/q1 = b2
b1∗q3/q1 = b3

Newton: "So the solutions Breton wrote down are indeed solutions to **x**•**A = b**

Einstein: "It's too simple. Let's do an example.

Breton: "All right. Let



A = **u1**∗**u1**+**u1**∗**u2**+**u1**∗**u3**
+2∗**u2**∗**u1**+2∗**u2**∗**u2**+2∗**u2**∗**u3**
+3∗**u3**∗**u1**+3∗**u3**∗**u2**+3∗**u3**∗**u3**

The null subset of **A** is {t1∗(2∗**u1−u2**)+t2∗(3∗**u1−u3**)} for any t1 and t2 in Q.

Let

b = u1+u2+u3

Then b1 = 1 = b2 = b3.

For **v** = **u1** we can calculate

q1 = v1∗a11+ v2∗a21+ v3∗a31 = 1
q2 = v1∗a12+ v2∗a22+ v3∗a32 = 1
q3 = v1∗a13+ v2∗a23+ v3∗a33 = 1

Then a solution to **x**•**A = b** with t1 = t2=1 is
x = (b1/q1)∗**v +**t1∗**n1 +**t2∗**n2**)
= **u1 +** 2∗**u1−u2 +**3∗**u1−u3**
=6∗**u1−u2 −u3**

since

(6∗**u1−u2 −u3**)•**A = u1 +u1+u1 = b**

All the equations yield the same set of solutions, namely a plane of vectors parallel to the null set of **A**.

Einstein: "Show me.”

Breton: "Easily. Suppose **x1** is a solution. Then **x1** added to any vector in **N**(**A**) is also a solution. It follows that the set

{**x1** +**N**(**A**)}

is a plane of vectors parallel to **N**(**A**).

Einstein: "Are they the only solutions? Perhaps other solutions exist beyond that plane.”

Breton: "No. If **x2** were any other solution, then **x1−x2** lies in the null set of **A** as we showed earlier.

Now can we find the solution with the least magnitude?

Newton: "Again, the set of solutions can be expressed as

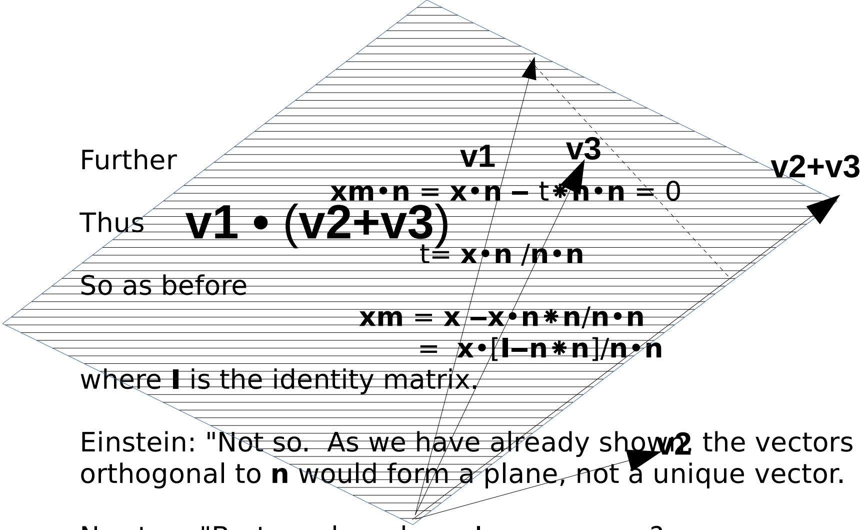
{**x**} = **xm** +{**y|y** is a vector in **N**(**A**)}

where **xm** is orthogonal to **N**(**A**). The solution **xm** would be the one with minimum magnitude.

Suppose now that both **n**, a vector in the null set, and some solution **x** are known. Then

xm = **x −** t∗**n**

for some t.



Further

$$\mathbf{xm}\bullet\mathbf{n} = \mathbf{x}\bullet\mathbf{n} - t*\mathbf{n}\bullet\mathbf{n} = 0$$

Thus

$$t = \mathbf{x}\bullet\mathbf{n} / \mathbf{n}\bullet\mathbf{n}$$

So as before

$$\mathbf{xm} = \mathbf{x} - \mathbf{x}\bullet\mathbf{n}*\mathbf{n}/\mathbf{n}\bullet\mathbf{n}$$
$$= \mathbf{x}\bullet[\mathbf{I}-\mathbf{n}*\mathbf{n}]/\mathbf{n}\bullet\mathbf{n}$$

where **I** is the identity matrix.

Einstein: "Not so. As we have already shown, the vectors orthogonal to **n** would form a plane, not a unique vector.

Newton: "Breton where have I gone wrong?

Breton: "Einstein is right. Why not choose *two* non-parallel vectors in **N**(**A**) and require **xm** to be orthogonal to both. Then **xm** will be unique.

Newton: "Something like a cross product. So let **n1** and **n2** both be non-parallel vectors of **N**(**A**). Then

$$\mathbf{xm}\bullet\mathbf{n1} = 0$$
$$\mathbf{xm}\bullet\mathbf{n2} = 0$$

Also suppose a solution x is known. Then

$$\mathbf{x} = \mathbf{xm} + t1*\mathbf{n1} + t2*\mathbf{n2}$$

for some t1 and t2. So

$$\mathbf{xm} = \mathbf{x} - (t1*\mathbf{n1} + t2*\mathbf{n2})$$
$$\mathbf{xm}\bullet\mathbf{n1} = \mathbf{x}\bullet\mathbf{n1} - (t1*\mathbf{n1}\bullet\mathbf{n1} + t2*\mathbf{n2}\bullet\mathbf{n1}) = 0$$
$$\mathbf{xm}\bullet\mathbf{n2} = \mathbf{x}\bullet\mathbf{n2} - (t1*\mathbf{n1}\bullet\mathbf{n2} + t2*\mathbf{n2}\bullet\mathbf{n2}) = 0$$

The two unknowns, t1 and t2 may thus be solved as

$$t1*\mathbf{n1}\bullet\mathbf{n1} + t2*\mathbf{n2}\bullet\mathbf{n1}) = \mathbf{x}\bullet\mathbf{n1}$$
$$t1*\mathbf{n1}\bullet\mathbf{n2} + t2*\mathbf{n2}\bullet\mathbf{n2}) = \mathbf{x}\bullet\mathbf{n2}$$

so

$$t1*\mathbf{n1}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} + t2*\mathbf{n2}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} = \mathbf{x}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2}$$
$$t1*\mathbf{n1}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1} + t2*\mathbf{n2}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1} = \mathbf{x}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1}$$

so subtracting

$$t1*\mathbf{n1}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} - t1*\mathbf{n1}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1}$$
$$= \mathbf{x}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} - \mathbf{x}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1}$$

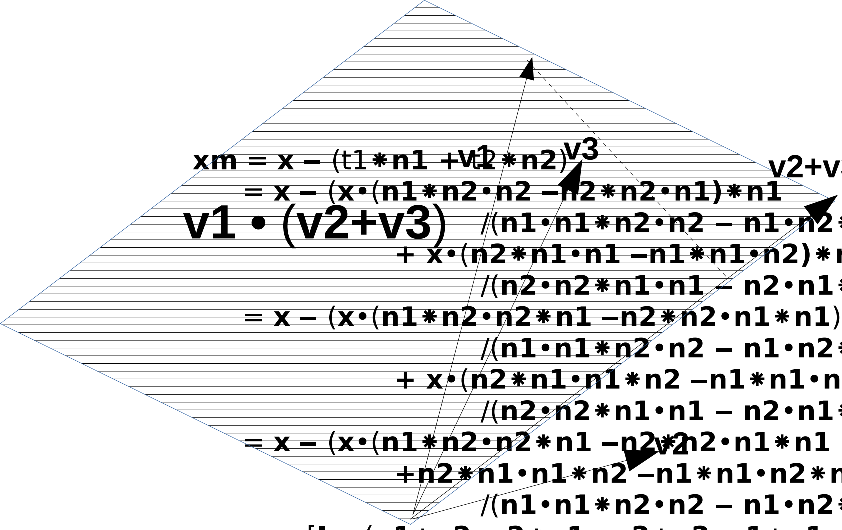
that is,

$$t1 = \mathbf{x}\bullet(\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} - \mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1})$$
$$/(\mathbf{n1}\bullet\mathbf{n1}*\mathbf{n2}\bullet\mathbf{n2} - \mathbf{n1}\bullet\mathbf{n2}*\mathbf{n2}\bullet\mathbf{n1})$$

Likewise

$$t2 = \mathbf{x}\bullet(\mathbf{n2}*\mathbf{n1}\bullet\mathbf{n1} - \mathbf{n1}*\mathbf{n1}\bullet\mathbf{n2})$$
$$/(\mathbf{n2}\bullet\mathbf{n2}*\mathbf{n1}\bullet\mathbf{n1} - \mathbf{n2}\bullet\mathbf{n1}*\mathbf{n1}\bullet\mathbf{n2})$$

Consequently,

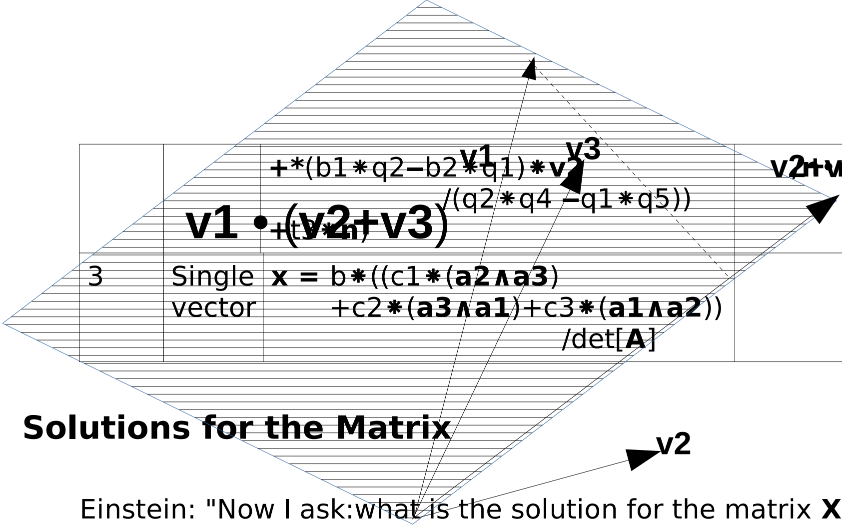


xm = **x −** (t1∗**n1** +t2∗**n2**)
    = **x −** (**x**•(**n1**∗**n2**•**n2** −**n2**∗**n2**•**n1**)∗**n1**
                /(**n1**•**n1**∗**n2**•**n2** − **n1**•**n2**∗**n2**•**n1**)
          + **x**•(**n2**∗**n1**•**n1** −**n1**∗**n1**•**n2**)∗**n2**
                /(**n2**•**n2**∗**n1**•**n1** − **n2**•**n1**∗**n1**•**n2**))
    = **x −** (**x**•(**n1**∗**n2**•**n2**∗**n1** −**n2**∗**n2**•**n1**∗**n1**)
                /(**n1**•**n1**∗**n2**•**n2** − **n1**•**n2**∗**n2**•**n1**)
          + **x**•(**n2**∗**n1**•**n1**∗**n2** −**n1**∗**n1**•**n2**∗**n2**)
                /(**n2**•**n2**∗**n1**•**n1** − **n2**•**n1**∗**n1**•**n2**))
    = **x −** (**x**•(**n1**∗**n2**•**n2**∗**n1** −**n2**∗**n2**•**n1**∗**n1**
          +**n2**∗**n1**•**n1**∗**n2** −**n1**∗**n1**•**n2**∗**n2**)
                /(**n1**•**n1**∗**n2**•**n2** − **n1**•**n2**∗**n2**•**n1**)
    = **x**•[**I −** (**n1**∗**n2**•**n2**∗**n1** −**n2**∗**n2**•**n1**∗**n1**
          +**n2**∗**n1**•**n1**∗**n2** −**n1**∗**n1**•**n2**∗**n2**)
                /(**n1**•**n1**∗**n2**•**n2** − **n1**•**n2**∗**n2**•**n1**)]
where again **I** is the identity matrix.

Breton: "If **n1** and **n2** are chosen orthogonalically
xm = **x**•[**I −** (**n2**•**n2**∗**n1**∗**n1** +**n1**•**n1**∗**n2**∗**n2**)
                /(**n1**•**n1**∗**n2**•**n2**)]
Newton would you construct a table of these solutions.

Newton: "Gladly.

Solutions of **x•A = b**, **A** and **b** given			
Rank	**N**(**A**)	Solutions	**xm**
0	**V3**	**0**	**0**
1	Plane	**x =** (b1/q1)∗**v** +t1∗**n1** +t2∗**n2**)	**xm** = (**x**•[**I** **−** (**n1**∗**n2** •**n2**∗**n1** −**n2**∗**n2** •**n1**∗**n1** +**n2**∗**n1** •**n1**∗**n2** −**n1**∗**n1** •**n2**∗**n2**)] /(**n1**•**n1** ∗**n2**•**n2** − **n1**•**n2** ∗**n2**•**n1**)
2	Line	**x =** ((b2∗q4−b1∗q5)∗**v1** /(q2∗q4 −q1∗q5))	**xm** = **x**•[**I** −(**n**∗**n**



		+*(b1∗q2−b2∗q1)∗**v2** /(q2∗q4 −q1∗q5)) +(t3∗m)	,n)]
3	Single vector	**x =** b∗((c1∗(**a2∧a3**) +c2∗(**a3∧a1**)+c3∗(**a1∧a2**)) /det[**A**]	

Solutions for the Matrix

Einstein: "Now I ask:what is the solution for the matrix **X** in the equation

v1•X = v2

where the **v1** and **v2** are given?

Breton: "I suspect rarely does a unique answer exist. Let us start with some definitions.

X = **u1∗x1+u2∗x2+u3∗x3**
x1 = x11∗**u1+**x12∗**u2+**x13∗**u3**
x2 = x21∗**u1+**x22∗**u2+**x23∗**u3**
x3 = x31∗**u1+**x32∗**u2+**x33∗**u3**
v1 = v11∗**u1+**v12∗**u2+**v13∗**u3**
v2 = v21∗**u1+**v22∗**u2+**v23∗**u3**
v1•X = (v11∗x11+v12∗x21+v13∗x31)∗**u1**
+ (v11∗x12+v12∗x22+v13∗x32)∗**u2**
+ (v11∗x13+v12∗x23+v13∗x33)∗**u3**

From these definitions, you can see the solution calls for determining nine unknowns, the xij, from three equations,

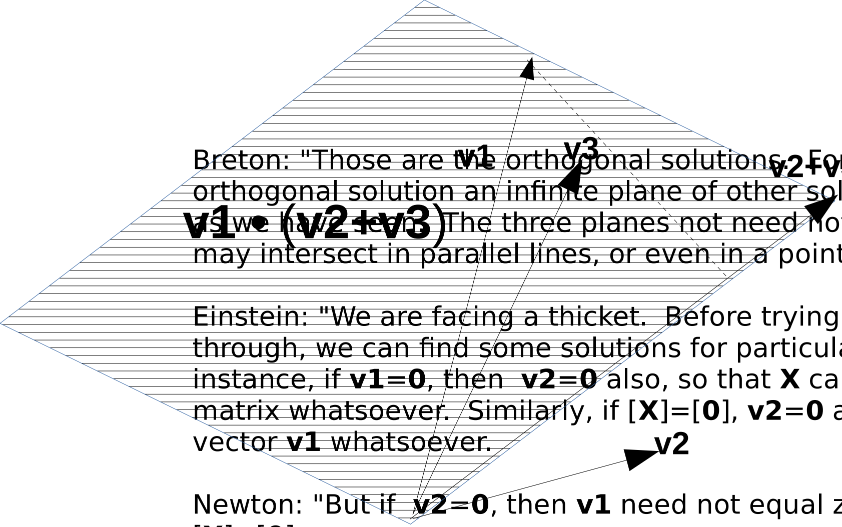
v1•(x11∗**u1∗u1**+x21∗**u2∗u1**+x31∗**u3∗u1**) **=** v21∗**u1**
v1•(x12∗**u1∗u2**+x22∗**u2∗u2**+x32∗**u3∗u2**) **=** v21∗**u2**
v1•(x13∗**u1∗u3**+x23∗**u2∗u3**+x33∗**u3∗u3**) **=** v21∗**u3**

that is,

v1•(x11∗**u1**+x21∗**u2**+x31∗**u3**) **=** v21
v1•(x12∗**u1**+x22∗**u2**+x32∗**u3**) **=** v22
v1•(x13∗**u1**+x23∗**u2**+x33∗**u3**) **=** v23

Newton: "We know the solutions to these equations. Remember the solution to **x•v1** = q1? These equations have the same form. So

(x11∗**u1**+x21∗**u2**+x31∗**u3**) = v21∗**qd**(**v1**)
(x12∗**u1**+x22∗**u2**+x32∗**u3**) = v22∗**qd**(**v1**)
(x13∗**u1**+x23∗**u2**+x33∗**u3**) = v23∗**qd**(**v1**)



Breton: "Those are the orthogonal solutions. For each orthogonal solution an infinite plane of other solutions exists, as we have seen. The three planes not need not intersect, may intersect in parallel lines, or even in a point.

Einstein: "We are facing a thicket. Before trying to cut through, we can find some solutions for particular cases. For instance, if **v1**=**0**, then **v2**=**0** also, so that **X** can be any matrix whatsoever. Similarly, if [**X**]=[**0**], **v2**=**0** also, for any vector **v1** whatsoever.

Newton: "But if **v2**=**0**, then **v1** need not equal zero, nor need [**X**]=[**0**].

Breton: "So the trivial solution for a matrix of rank 0 is known. Shall we try for solutions for a matrix of rank3?

Newton: "Which is to say that the matrix has an inverse.

Breton: "Right. If **X** has an inverse, then

$$\mathbf{X}^{-1} = \mathbf{x2 \wedge x3 * u1 + x3 \wedge x1 * u2 + x1 \wedge x2 * u3}/\det(\mathbf{X})$$

and

$$\mathbf{v1 = v2 \bullet X}^{-1}$$

Einstein: "Is det(**X**) equal to det($\mathbf{X}^{-1}$)?

Breton: "A good question Einstein. If so, then $\mathbf{X}^{-1}$ is also a matrix of rank3. We know that the determinant is a scalar triple product, namely,

$$\begin{aligned}\det(\mathbf{X}) &= \mathbf{x1 \bullet (x2 \wedge x3)}\\ &= x11 * x22 * x33\\ &\quad +x12 * x23 * x31\\ &\quad +x13 * x21 * x32\\ &\quad -x11 * x23 * x32\\ &\quad -x12 * x21 * x33\\ &\quad -x13 * x22 * x31\end{aligned}$$

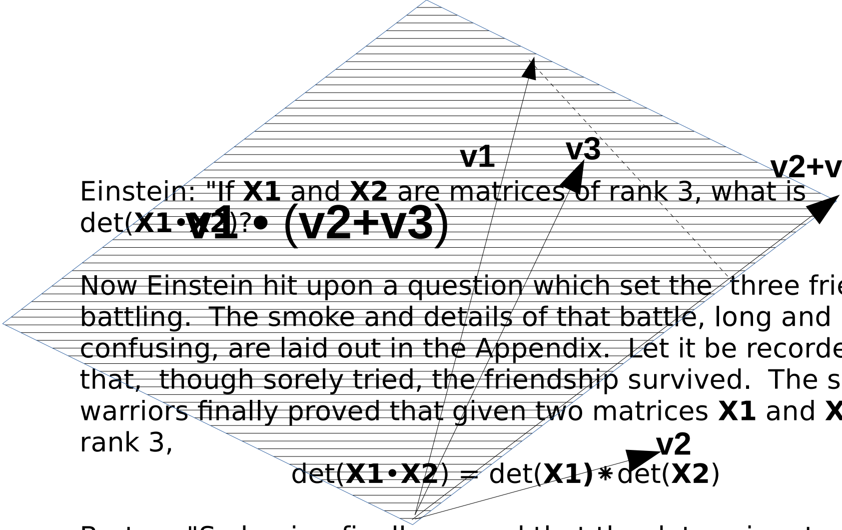
Newton: "So det(**X**)=det(**T**[(**X**]).

Breton: "Well yes, but we are looking for the determinant of the inverse.

$$\mathbf{X}^{-1} = ((\mathbf{x2 \wedge x3) * u1 + (x3 \wedge x1) * u2 + (x1 \wedge x2) * u3})/\det(\mathbf{X})$$

and its transpose

$$\mathbf{T}[\mathbf{X}^{-1}] = (\mathbf{u1 * (x2 \wedge x3) + u2 * (x3 \wedge x1) + u3 * (x1 \wedge x2)})/\det(\mathbf{X}).$$



Einstein: "If **X1** and **X2** are matrices of rank 3, what is det(**X1•X2**)?

Now Einstein hit upon a question which set the three friends battling. The smoke and details of that battle, long and confusing, are laid out in the Appendix. Let it be recorded that, though sorely tried, the friendship survived. The scarred warriors finally proved that given two matrices **X1** and **X2** of rank 3,

det(**X1•X2**) = det(**X1**)∗det(**X2**)

Breton: "So having finally proved that the determinant of the product of two matrices of rank 3 equals the product of the determinant of each matrix, we come to a easy answer to our earlier question: What is the determinant of $\mathbf{X}^{-1}$?

Newton: " We do? Show us.

Breton: "Given a matrix **X** of rank three, then as you observed, Newton, it has an inverse, that is,

$$\mathbf{X} \bullet \mathbf{X}^{-1} = \mathbf{I}$$

So

$$\det(\mathbf{X} \bullet \mathbf{X}^{-1}) = \det(\mathbf{X}) * \det(\mathbf{X}^{-1}) = \det(\mathbf{I})$$

Therefore,

$$\det(\mathbf{X}^{-1}) = \det(\mathbf{I})/\det(\mathbf{X})$$

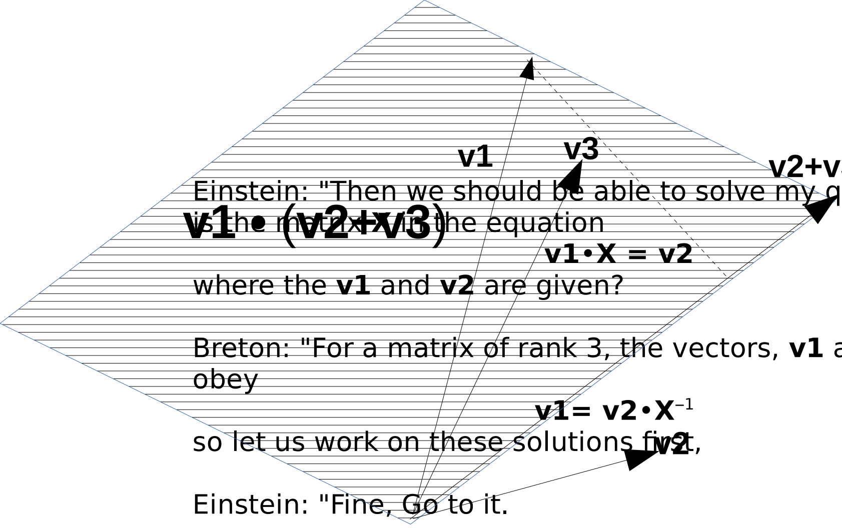
Newton: "And what is det(**I**)?

Einstein: "You can easily calculate det(**I**)= 1.

Newton: "This set of determinants has inverses, like quotient numbers—and like quotient vectors. Have we, in fact, defined an algebra of matrices?

Breton: "Just so. If **X** is a matrix of rank 3, then $\det(\mathbf{X}^{-1})$ is non-zero which means that $\mathbf{X}^{-1}$ is itself a matrix of rank 3. With the inclusion of inverse matrices, the set of matrices of rank 3 do indeed constitute an algebra since given any two such matrices, **X1 and X2** ,

X1 + X2 is defined,
X1 – X2 is defined,
X1 • X2 is defined,
X1 / **X2** = **X1 • X2** $^{-1}$ is defined,



Einstein: "Then we should be able to solve my question:what is the matrix X in the equation

v1•X = v2

where the **v1** and **v2** are given?

Breton: "For a matrix of rank 3, the vectors, **v1** and **v2** must obey

v1= v2•X^{-1}

so let us work on these solutions first,

Einstein: "Fine, Go to it.

Breton: "First, I suspect there exist more than one solution. So let us start by finding at least one solution. Suppose **X** diagonal. Then let

v1= v11✱**u1 +** v12✱**u2 +**v13✱**u3**
v2= v21✱**u1 +** v22✱**u2 +**v23✱**u3**
X= g1✱**u1**✱**u1 +**g2✱**u2**✱**u2 +**g3✱**u3**✱**u3**

If we can find the g's in terms of the v's we will have a solution. So expanding **v1•X = v2** we have

v11✱**u1 +** v12✱**u2 +**v13✱**u3**
•[g1✱**u1**✱**u1 +**g2✱**u2**✱**u2 +**g3✱**u3**✱**u3**]
= v21✱**u1 +** v22✱**u2 +**v23✱**u3**

that is,

v11✱g1✱**u1 +** v12✱g2✱**u2 +**v13✱g3✱**u3**
= v21✱**u1 +** v22✱**u2 +**v23✱**u3**

so

g1 = v21/v11
g2 = v22/v12
g3 = v23/v13

You can see easily that

X= v21✱**u1**✱**u1**/v11 **+** v22✱**u2**✱**u2**/v12 **+**v23✱**u3**✱**u3**/v13

is a solution.

Newton: "How about the inverse.

Breton: "For the diagonal case

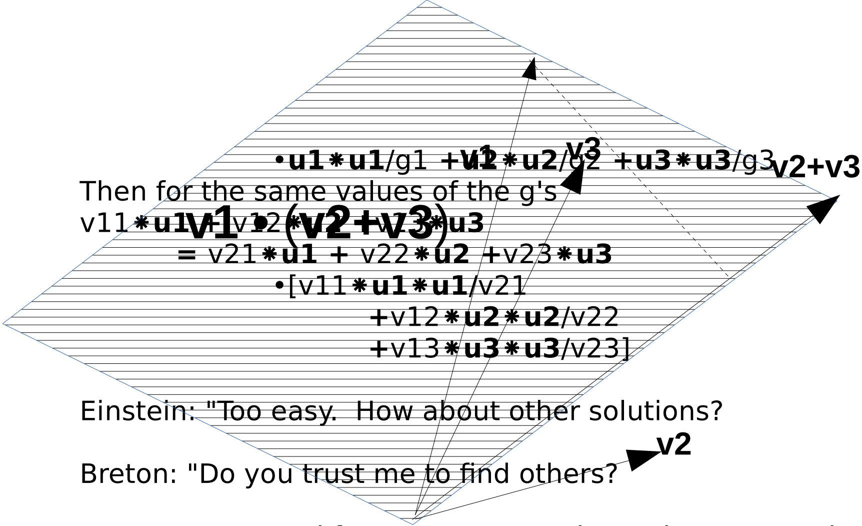
X^{-1} = **u1**✱**u1**/g1 **+u2**✱**u2**/g2 **+u3**✱**u3**/g3

and

v1= v2•X^{-1}

expands as

v11✱**u1 +** v12✱**u2 +**v13✱**u3**
= v21✱**u1 +** v22✱**u2 +**v23✱**u3**



•**u1∗u1**/g1 +**u2∗u2**/g2 +**u3∗u3**/g3

Then for the same values of the g's
v11∗**u1** + v12∗**u2** + v13∗**u3**
= v21∗**u1 +** v22∗**u2 +**v23∗**u3**
•[v11∗**u1∗u1**/v21
+v12∗**u2∗u2**/v22
+v13∗**u3∗u3**/v23]

Einstein: "Too easy. How about other solutions?

Breton: "Do you trust me to find others?

Newton: "No need for trust. Just produce what you say is a solution; we can easily verify it.

Breton: "All right. I will use a method for finding rank3 solutions which can be modified to find rank2 and rank1 solutions also.
To find the solutions of **v1•A** = **v2** of rank 3 in general, chose three distinct directions, **ua1**, **ua2**, and **ua3**. Next, define

t1 ≡ **v2•u1**/(**v1•ua1**)
t2 ≡ **v2•u2**/(**v1•ua2**)
t3 ≡ **v2•u3**/(**v1•ua3**)

Then

A= t1∗**ua1∗u1** + t2∗**ua2∗u2** + t3∗**ua3∗u3**

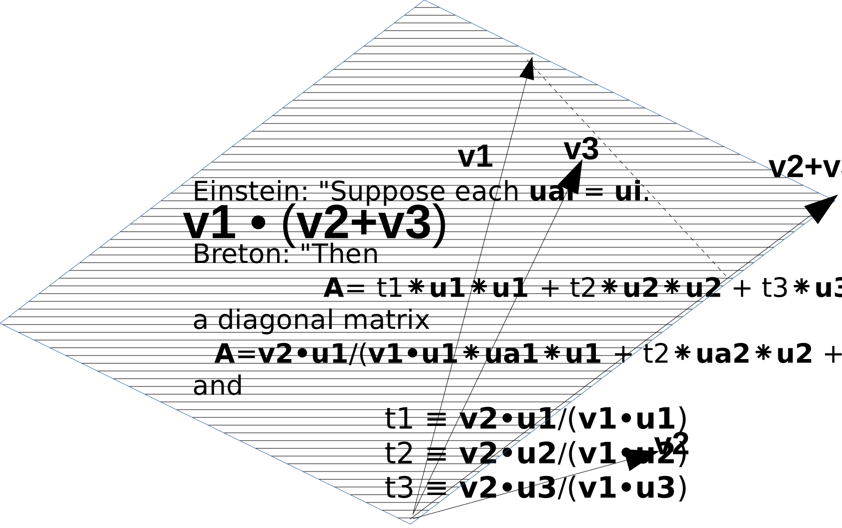
is a solution, since

v1•**A** = **v1**•[t1∗**ua1∗u1**
+ t2∗**ua2∗u2** + t3∗**ua3∗u3**]
= t1∗**v1**•**ua1∗u1**
+ t2∗**v1**•**ua2∗u2** + t3∗**v1**•**ua3∗u3**
= (**v2•u1**/**v1•ua1**)∗**v1**•**ua1∗u1**
+ (**v2•u2**/**v1•ua2**)∗**v1**•**ua2∗u2**
+ (**v2•u3**/**v1•ua3**)∗**v1**•**ua3∗u3**
= **v2•**[**u1∗u1 + u2∗u2 + u3∗u3**]
= **v2**

Newton: "That's like magic. How did you know how to define the t's?

Breton: "You didn't trust me! Still the solution is verified.

Newton: "So an infinity of choices is available, for all the different choices of **ua**'s possible.



Einstein: "Suppose each **uai = ui**.

Breton: "Then

A= t1∗**u1∗u1** + t2∗**u2∗u2** + t3∗**u3∗u3**

a diagonal matrix

A=v2•u1/(**v1•u1∗ua1∗u1** + t2∗**ua2∗u2** + t3∗**ua3∗u3**

and

t1 ≡ **v2•u1**/(**v1•u1**)
t2 ≡ **v2•u2**/(**v1•u2**)
t3 ≡ **v2•u3**/(**v1•u3**)

Newton: "Which is precisely our former solution.

Einstein: "How about the rank2 solutions?

To find the rank 2 solutions of **v1•A** = **v2** chose two distinct directions, **ua1**, **ua2**, Next define

t1 ≡ **v2•u1/v1•ua1**
t2 ≡ **v2•u2/v1•ua2**
t3 ≡ **v2•u3/v1•ua2**

Then

A= t1∗**ua1∗u1** + t2∗**ua2∗u2** + t3∗**ua2∗u3**

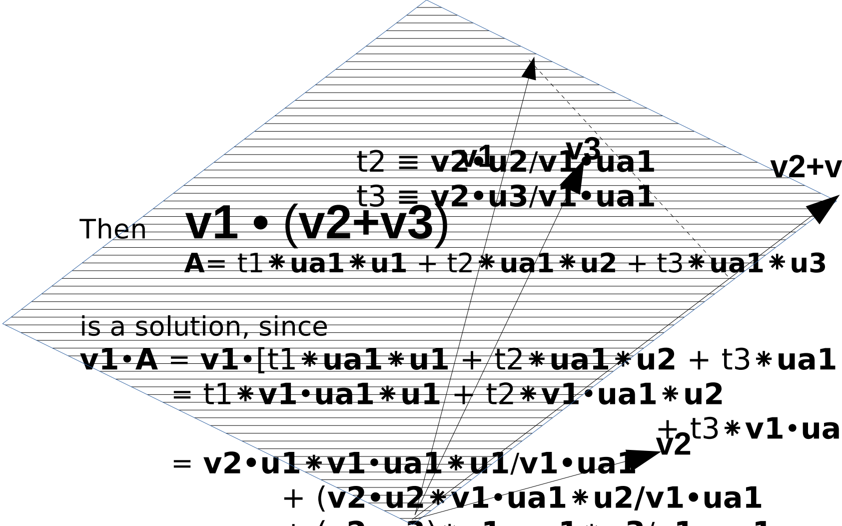
is a solution, since

v1•A = **v1**•[t1∗**ua1∗u1** + t2∗**ua2∗u2** + t3∗**ua2∗u3**]
= t1∗**v1•ua1∗u1** + t2∗**v1•ua2∗u2**
+ t3∗**v1•ua2∗u3**
= (**v2•u1∗v1•ua1∗u1/v1•ua1)**
+ (**v2•u2∗v1•ua2∗u2/v1•ua2**)
+ (**v2•u3∗v1•ua2∗u3/v1•ua2)**
= **v2**•[**u1∗u1 + u2∗u2 + u3∗u3**]
= **v2**

Six similar variations may be formed for any arbitrary choice of any two distinct directions.

Newton: "So the method for the magic is clear. Let me try the rank 1 solutions of **v1•A** = **v2**. Chose any direction **ua1** not orthogonal to **v1**. Next, define

t1 ≡ **v2•u1/v1•ua1**



t2 ≡ **v2•u2/v1•ua1**
t3 ≡ **v2•u3/v1•ua1**

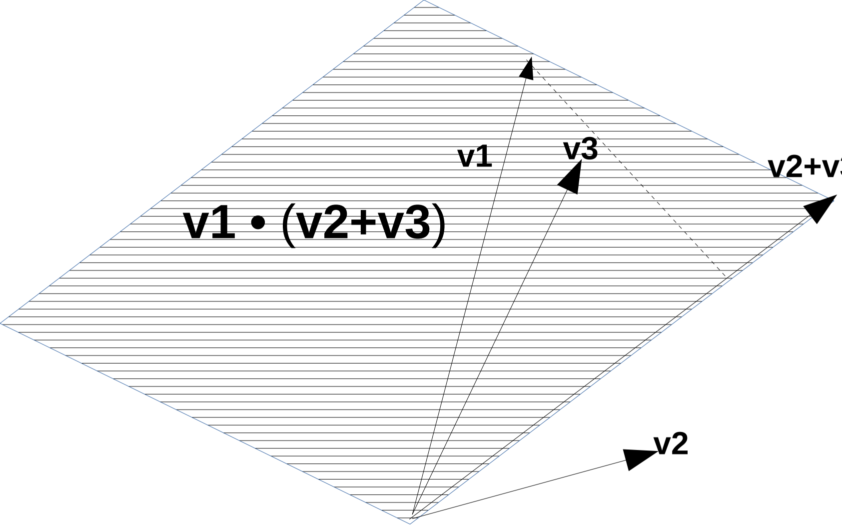
Then
A= t1∗**ua1**∗**u1** + t2∗**ua1**∗**u2** + t3∗**ua1**∗**u3**

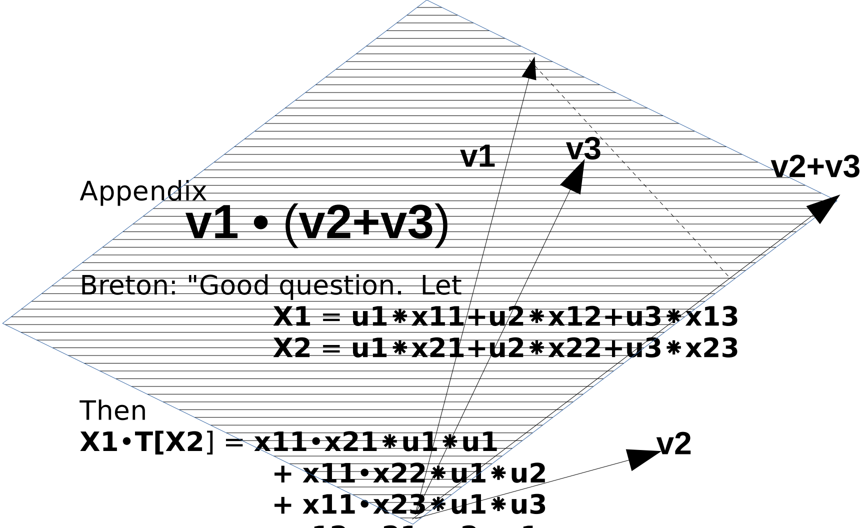
is a solution, since
v1•**A** = **v1**•[t1∗**ua1**∗**u1** + t2∗**ua1**∗**u2** + t3∗**ua1**∗**u3**]
= t1∗**v1**•**ua1**∗**u1** + t2∗**v1**•**ua1**∗**u2**
+ t3∗**v1**•**ua1**∗**u3**
= **v2•u1**∗**v1**•**ua1**∗**u1/v1•ua1**
+ (**v2•u2**∗**v1**•**ua1**∗**u2/v1•ua1**
+ (**v2•u3**)∗**v1**•**ua1**∗**u3/v1•ua1**
= **v2•**[**u1**∗**u1 + u2**∗**u2 + u3**∗**u3**]
= **v2**

Breton: " Suppose **ua1** = **uv1**

Newton: "Then
A = t1∗**uv1**∗**u1** + t2∗**uv1**∗**u2** + t3∗∗**uv1 u3**
= (**v2•u1**)∗**u1**∗**uv1**/v1
+ (**v2•u2**)∗**u2**∗**uv1**/v1
+ (**v2•u3**)∗**u3**∗**v1**/v1
= **v2•**(**u1**∗**u1**+ **u2**∗**u2** + **u3**∗**u3**)∗**uv1**/v1
= **v2**∗**uv1**/v1.
Breton: "which can be rewritten
A = = **v2**∗**qd**(**v1**),
an outer product.
Fuvthevmove fov any **v** and a matrix **A** of rank 1 which solves the equation **v1**•**A** = **v2**

v•**A** = **v**•[t1∗**u1**∗**ua1** + t2∗**u2**∗**ua1** + t3∗**u3**∗**ua1**]
= t1∗**v**•**ua1**∗**u1** + t2∗**v**•**ua1**∗**u2**
+ t3∗**v**•**ua1**∗**u3**
= **v2•u1**∗**v**•**ua1**∗**u1/v1•ua1**
+ (**v2•u2**∗**v**•**ua1**∗**u2/v1•ua1**
+ (**v2•u3**∗**v**•**ua1**∗**u3/v1•ua1**
= (**v•ua1/v1•ua1**)
∗**v2•**[**u1**∗**u1 + u2**∗**u2 + u3**∗**u3**]
= (**v•ua1/v1•ua1**)∗**v2**
Thus **A** maps any vector into a unique direction.


v1
v3
v2+v3
v1 • (v2+v3)
v2



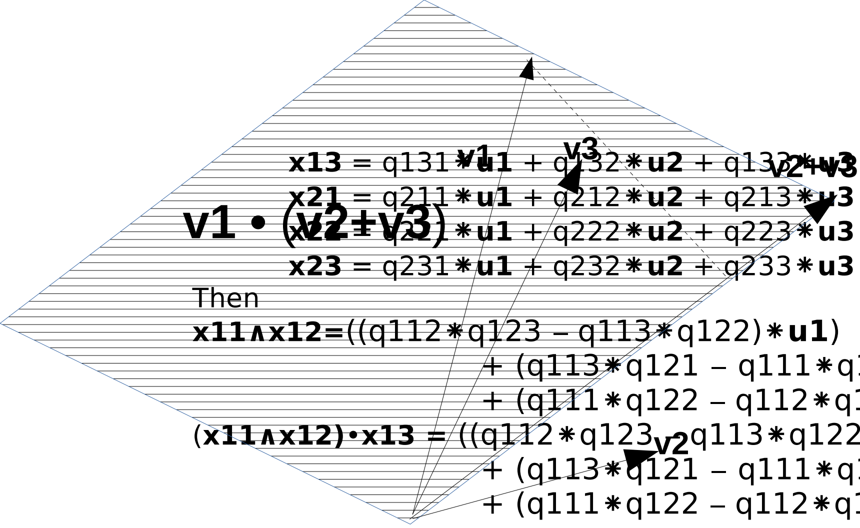
Appendix

Breton: "Good question. Let
X1 = u1∗x11+u2∗x12+u3∗x13
X2 = u1∗x21+u2∗x22+u3∗x23

Then
X1•T[X2] = **x11•x21∗u1∗u1**
+ x11•x22∗u1∗u2
+ x11•x23∗u1∗u3
+x12•x21∗u2∗u1
+ x12•x22∗u2∗u2
+ x12•x23∗u2∗u3
+ x13•x21∗u3∗u1
+ x13•x22∗u3∗u2
+ x13•x23∗u3∗u3
So
det(**X1•T[X2**])
= (**x11•x21∗u1 + x11•x22∗u2 + x11•x23∗u3**)
∧(**x12•x21∗u1+x12•x22∗u2+x12•x23∗u3**)
•(x13•x21∗u1+x13•x22∗u2+x13•x23∗u3)
=(**x11•x22∗x12•x23− x11•x23∗x12•x22)∗u1**
+(x11•x23∗x12•x21−x11•x21∗x12•x23)**∗u2**
+(**x11•x21∗x12•x22−x11•x22∗x12•x21)∗u3**)
•(x13•x21∗u1+x13•x22∗u2+x13•x23∗u3)
=**x11•x22∗x12•x23∗x13•x21**
− x11•x23∗x12•x22∗x13•x21
+**x11•x23∗x12•x21∗x13•x22**
−x11•x21∗x12•x23∗x13•x22
+**x11•x21∗x12•x22∗x13•x23**
−x11•x22∗x12•x21∗x13•x23
Wouldn't it be remarkable if this porridge of symbols equaled det(**X1)**∗det(**X2**)?

Einstein: "Remark away.

Breton: "All right.
det(**X1**)= (**x11∧x12)•x13**
det(**X2**)= (**x21∧x22)•x23**
Let
x11 = q111∗**u1** + q112∗**u2** + q113∗**u3**
x12 = q121∗**u1** + q122∗**u2** + q123∗**u3**



$\mathbf{x13} = q131 \ast \mathbf{u1} + q132 \ast \mathbf{u2} + q133 \ast \mathbf{u3}$

$\mathbf{x21} = q211 \ast \mathbf{u1} + q212 \ast \mathbf{u2} + q213 \ast \mathbf{u3}$

$\mathbf{x22} = q221 \ast \mathbf{u1} + q222 \ast \mathbf{u2} + q223 \ast \mathbf{u3}$

$\mathbf{x23} = q231 \ast \mathbf{u1} + q232 \ast \mathbf{u2} + q233 \ast \mathbf{u3}$

Then

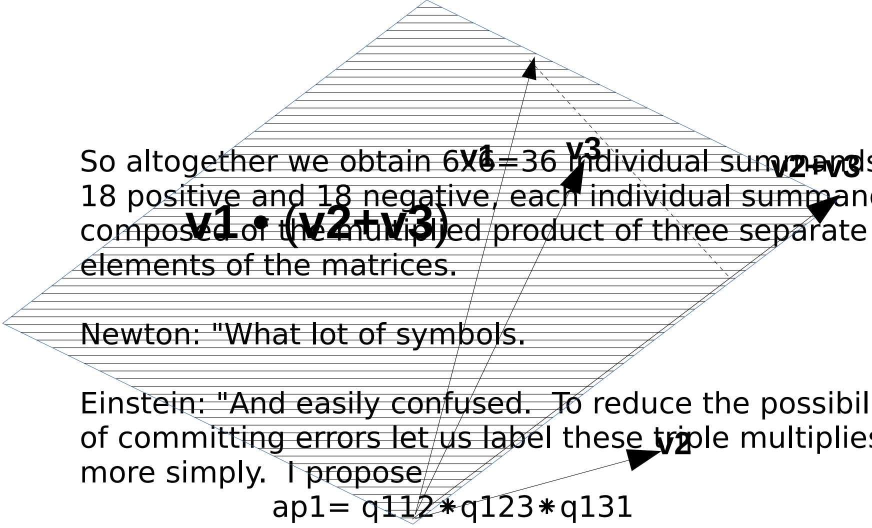
$\mathbf{x11 \wedge x12 =} ((q112 \ast q123 - q113 \ast q122) \ast \mathbf{u1})$
$+ (q113 \ast q121 - q111 \ast q123) \ast \mathbf{u2}$
$+ (q111 \ast q122 - q112 \ast q121) \ast \mathbf{u3})$

$(\mathbf{x11 \wedge x12) \bullet x13} = ((q112 \ast q123 - q113 \ast q122) \ast \mathbf{u1})$
$+ (q113 \ast q121 - q111 \ast q123) \ast \mathbf{u2}$
$+ (q111 \ast q122 - q112 \ast q121) \ast \mathbf{u3})$
$\bullet \mathbf{(}q131 \ast \mathbf{u1} + q132 \ast \mathbf{u2} + q133 \ast \mathbf{u3)}$
$\mathbf{=} ((q112 \ast q123 - q113 \ast q122) \ast q131$
$+ (q113 \ast q121 - q111 \ast q123) \ast q132$
$+ (q111 \ast q122 - q112 \ast q121) \ast q133$

$\mathbf{(x21 \wedge x22) \bullet x23 =}$
$\mathbf{=} ((q212 \ast q223 - q213 \ast q222) \ast q231$
$+ (q213 \ast q221 - q211 \ast q223) \ast q232$
$+ (q211 \ast q222 - q212 \ast q221) \ast q233$

So

$\det(\mathbf{X1)} \ast \det(\mathbf{X2})$
$=(\mathbf{x11 \wedge x12) \bullet x13 \ast (x21 \wedge x22) \bullet x23}$
$\mathbf{=} ((q112 \ast q123 - q113 \ast q122) \ast q131$
$+ \quad + (q113 \ast q121 - q111 \ast q123) \ast q132$
$+ (q111 \ast q122 - q112 \ast q121) \ast q133$
$\ast ((q212 \ast q223 - q213 \ast q222) \ast q231$
$+ (q213 \ast q221 - q211 \ast q223) \ast q232$
$+ (q211 \ast q222 - q212 \ast q221) \ast q233$
$\mathbf{=} (q112 \ast q123 \ast q131 - q113 \ast q122 \ast q131$
$+ q113 \ast q121 \ast q132 - q111 \ast q123 \ast q132$
$+ (q111 \ast q122 \ast q133 - q112 \ast q121 \ast q133)$
$\ast (q212 \ast q223 \ast q231 - q213 \ast q222 \ast q231$
$+ (q213 \ast q221 \ast q232 - q211 \ast q223 \ast q232$
$+ (q211 \ast q222 \ast q233 - q212 \ast q221 \ast q233)$
$\mathbf{=} (q112 \ast q123 \ast q131 - q113 \ast q122 \ast q131$
$+ q113 \ast q121 \ast q132 - q111 \ast q123 \ast q132$
$+ q111 \ast q122 \ast q133 - q112 \ast q121 \ast q133)$
$\ast (q212 \ast q223 \ast q231 - q213 \ast q222 \ast q231$
$+ (q213 \ast q221 \ast q232 - q211 \ast q223 \ast q232$
$+ (q211 \ast q222 \ast q233 - q212 \ast q221 \ast q233)$



So altogether we obtain 6X6=36 individual summands, 18 positive and 18 negative, each individual summand composed of the multiplied product of three separate elements of the matrices.

Newton: "What lot of symbols.

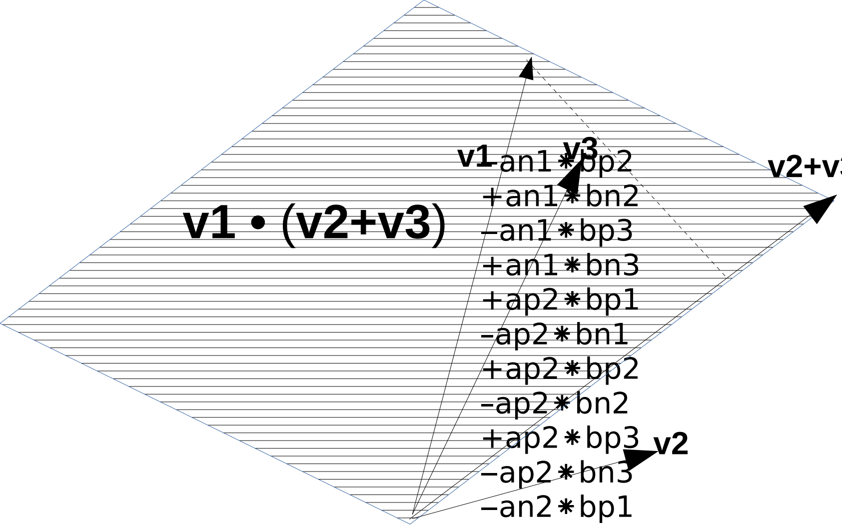
Einstein: "And easily confused. To reduce the possibility of committing errors let us label these triple multiplies more simply. I propose

ap1= q112∗q123∗q131
an1= q113∗q122∗q131
ap2= q113∗q121∗q132
an2= q111∗q123∗q132
ap3= q111∗q122∗q133
an3= q112∗q121∗q133
bp1= q212∗q223∗q231
bn1= q213∗q222∗q231
bp2= q213∗q221∗q232
bn2= q211∗q223∗q232
bp3= q211∗q222∗q233
bn3= q212∗q221∗q233

Then in this new notation

det(**X1)**∗det(**X2**)
=(ap1 – an1 + ap2 – an2 + ap3 – an3)
∗ (bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
= ap1∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
−an1∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
+ ap2∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
−an2∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
+ap3∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)
– an3∗(bp1 – bn1 + bp2 – bn2 + bp3 – bn3)

det(**X1)**∗det(**X2**) = ap1∗bp1
−ap1∗bn1
+ap1∗bp2
–ap1∗bn2
+ap1∗bp3
–ap1∗bn3
−an1∗bp1
+an1∗bn1



–an1∗bp2
+an1∗bn2
–an1∗bp3
+an1∗bn3
+ap2∗bp1
–ap2∗bn1
+ap2∗bp2
–ap2∗bn2
+ap2∗bp3
–ap2∗bn3
–an2∗bp1
+an2∗bn1
–an2∗bp2
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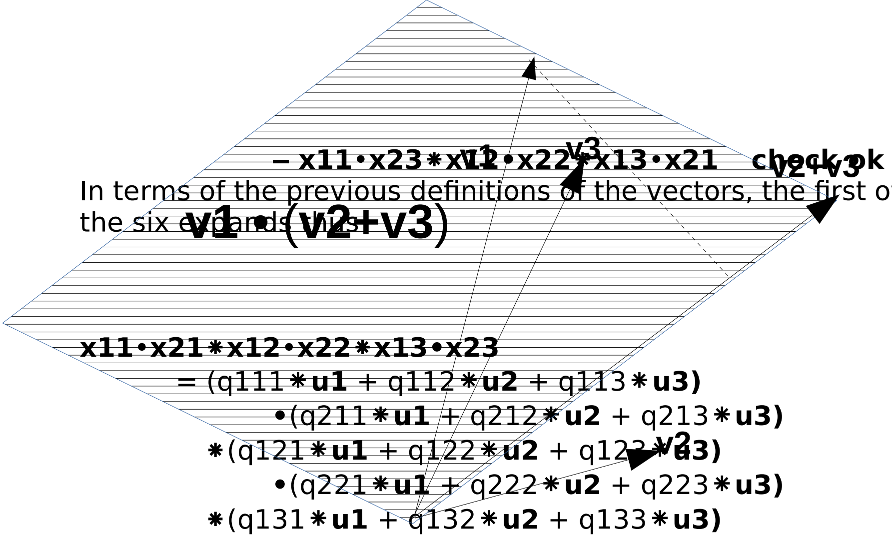
+ap3∗bp1
–ap3∗bn1
+ap3∗bp2
–ap3∗bn2
+ap3∗bp3
–ap3∗bn3
–an3∗bp1
+an3∗bn1
–an3∗bp2
+an3∗bn2
–an3∗bp3
+an3∗bn3

Einstein: "The thicket is somewhat thinned, but still formidable. You will have to prove that det(**X1•X2**) equals the same sum.

Breton: "Remember we noted det(**X1**) =det(**T**[**X1**]). So expanding the determinant of the multiplied matrices from above det(**X1•T[X2**])

=**x11•x21∗x12•x22∗x13•x23**
+ **x11•x22∗x12•x23∗x13•x21**
+ **x11•x23∗x12•x21∗x13•x22**
–x11•x21∗x12•x23∗x13•x22
–x11•x22∗x12•x21∗x13•x23



– x11•x23∗x12•x22∗x13•x21   check ok

In terms of the previous definitions of the vectors, the first of the six expands thus

x11•x21∗x12•x22∗x13•x23

= (q111∗**u1** + q112∗**u2** + q113∗**u3)**
•(q211∗**u1** + q212∗**u2** + q213∗**u3)**
∗(q121∗**u1** + q122∗**u2** + q123∗**u3)**
•(q221∗**u1** + q222∗**u2** + q223∗**u3)**
∗(q131∗**u1** + q132∗**u2** + q133∗**u3)**
•(q231∗**u1** + q232∗**u2** + q233∗**u3)**
= (q111∗q211 + q112∗q212 + q113∗q213**)**
∗(q121∗q221 + q122∗q222 + q123∗q223**)**
∗(q131∗q231 + q132∗q232 + q133∗q233**)**

So here we have 27 addends each composed of two matrix elements multiplied together.

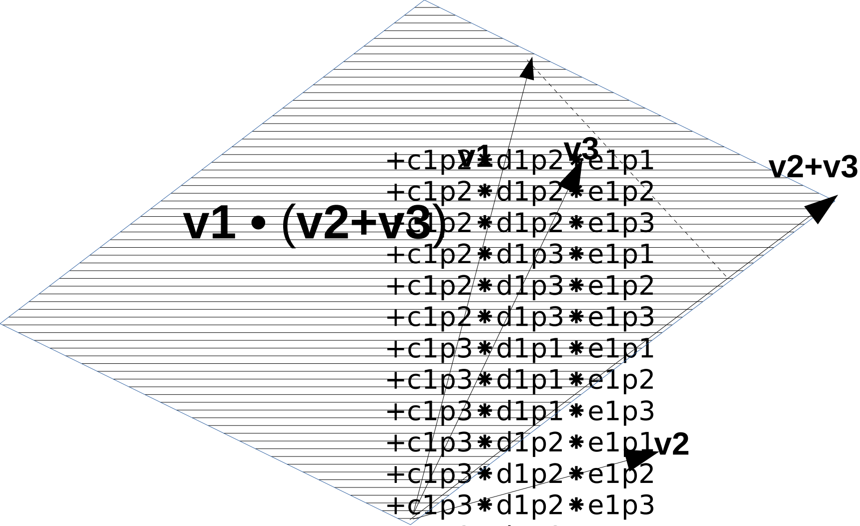
Let us label these double multiplies more simply to reduce the possibility of committing errors. I propose

c1p1 = q111∗q211
c1p2 = q112∗q212
c1p3 = q113∗q213
d1p1 = q121∗q221
d1p2 = q122∗q222
d1p3 = q123∗q223
e1p1 = q131∗q231
e1p2 = q132∗q232
e1p3 = q133∗q233

Then in terms of this new notation

x11•x21∗x12•x22∗x13•x23

= c1p1∗d1p1∗e1p1
+c1p1∗d1p1∗e1p2
+c1p1∗d1p1∗e1p3
+c1p1∗d1p2∗e1p1
+c1p1∗d1p2∗e1p2
+c1p1∗d1p2∗e1p3
+c1p1∗d1p3∗e1p1
+c1p1∗d1p3∗e1p2
+c1p1∗d1p3∗e1p3
+c1p2∗d1p1∗e1p1
+c1p2∗d1p1∗e1p2
+c1p2∗d1p1∗e1p3



+c1p2∗d1p2∗e1p1
+c1p2∗d1p2∗e1p2
+c1p2∗d1p2∗e1p3
+c1p2∗d1p3∗e1p1
+c1p2∗d1p3∗e1p2
+c1p2∗d1p3∗e1p3
+c1p3∗d1p1∗e1p1
+c1p3∗d1p1∗e1p2
+c1p3∗d1p1∗e1p3
+c1p3∗d1p2∗e1p1
+c1p3∗d1p2∗e1p2
+c1p3∗d1p2∗e1p3
+c1p3∗d1p3∗e1p1
+c1p3∗d1p3∗e1p2
+c1p3∗d1p3∗e1p3

Einstein: "So for this first addend expands into 27 addends each with six matrix elements multiplied together. Breton, it looks like you're done for.

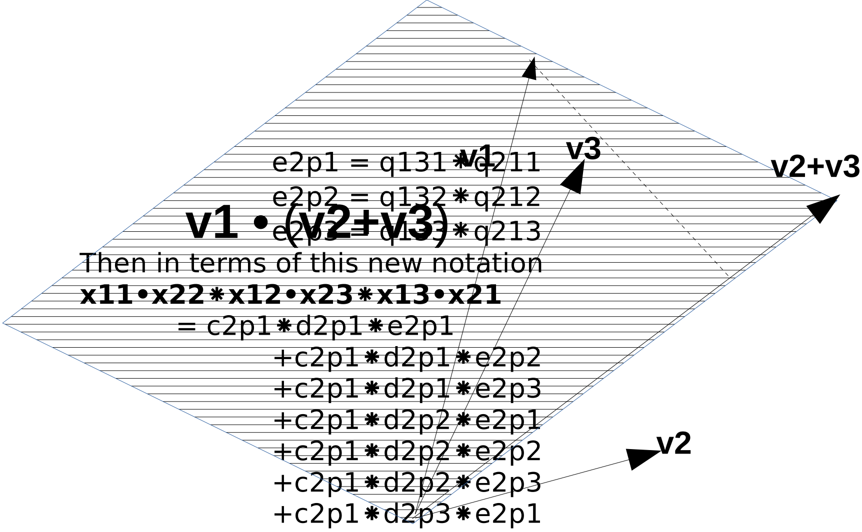
Breton: "Patience. Each of the c∗d∗e addends consists of 3∗2=6 elements of the original matrices while each of the a∗b addends consists of 2∗3=6 elements of the original matrices--an encouraging sign.

Let us continue to the second factor. For
x11•x22∗x12•x23∗x13•x21
= (q111∗**u1** + q112∗**u2** + q113∗**u3)**
•(q221∗**u1** + q222∗**u2** + q223∗**u3)**
∗(q121∗**u1** + q122∗**u2** + q123∗**u3)**
•(q231∗**u1** + q232∗**u2** + q233∗**u3)**
∗(q131∗**u1** + q132∗**u2** + q133∗**u3)**
•(q211∗**u1** + q212∗**u2** + q213∗**u3)**
= (q111∗q221 + q112∗q222 + q113∗q223**)**
∗(q121∗q231 + q122∗ q232 + q123∗q233**)**
∗(q131∗ q211 + q132∗ q212 + q133∗q213**)**

Again let us label these double multiplies more simply to reduce the possibility of committting errors. I propose

c2p1 = q111∗q221
c2p2 = q112∗q222
c2p3 = q113∗q223
d2p1 = q121∗q231
d2p2 = q122∗q232
d2p3 = q123∗q233

e2p1 = q131∗q211
e2p2 = q132∗q212
e2p3 = q133∗q213



Then in terms of this new notation

x11•x22∗x12•x23∗x13•x21

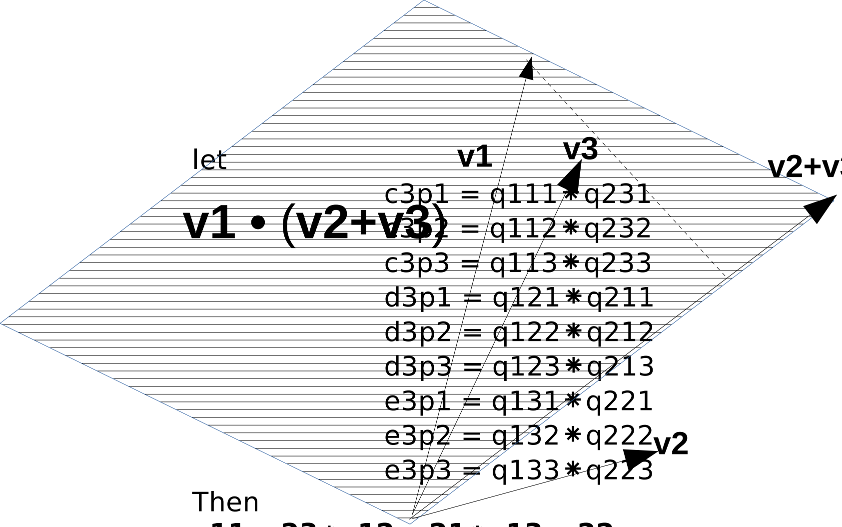
= c2p1∗d2p1∗e2p1
+c2p1∗d2p1∗e2p2
+c2p1∗d2p1∗e2p3
+c2p1∗d2p2∗e2p1
+c2p1∗d2p2∗e2p2
+c2p1∗d2p2∗e2p3
+c2p1∗d2p3∗e2p1
+c2p1∗d2p3∗e2p2
+c2p1∗d2p3∗e2p3
+c2p2∗d2p1∗e2p1
+c2p2∗d2p1∗e2p2
+c2p2∗d2p1∗e2p3
+c2p2∗d2p2∗e2p1
+c2p2∗d2p2∗e2p2
+c2p2∗d2p2∗e2p3
+c2p2∗d2p3∗e2p1
+c2p2∗d2p3∗e2p2
+c2p2∗d2p3∗e2p3
+c2p3∗d2p1∗e2p1
+c2p3∗d2p1∗e2p2
+c2p3∗d2p1∗e2p3
+c2p3∗d2p2∗e2p1
+c2p3∗d2p2∗e2p2
+c2p3∗d2p2∗e2p3
+c2p3∗d2p3∗e2p1
+c2p3∗d2p3∗e2p2
+c2p3∗d2p3∗e2p3

For the addend

x11•x23∗x12•x21∗x13•x22

= (q111∗**u1** + q112∗**u2** + q113∗**u3)**
•(q231∗**u1** + q232∗**u2** + q233∗**u3)**
∗(q121∗**u1** + q122∗**u2** + q123∗**u3)**
•(q211∗**u1** + q212∗**u2** + q213∗**u3)**
∗(q131∗**u1** + q132∗**u2** + q133∗**u3)**
•(q221∗**u1** + q222∗**u2** + q223∗**u3)**

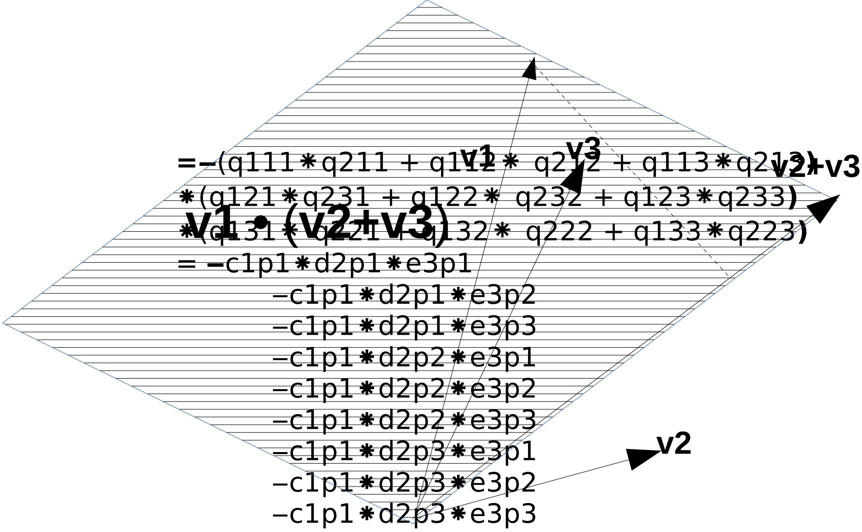
= (q111∗ q231 + q112∗ q232 + q113∗q233**)**
∗(q121∗ q211 + q122∗ q212 + q123∗q213**)**
∗(q131∗ q221 + q132∗ q222 + q133∗ q223**)**



let
c3p1 = q111✱q231
c3p2 = q112✱q232
c3p3 = q113✱q233
d3p1 = q121✱q211
d3p2 = q122✱q212
d3p3 = q123✱q213
e3p1 = q131✱q221
e3p2 = q132✱q222
e3p3 = q133✱q223

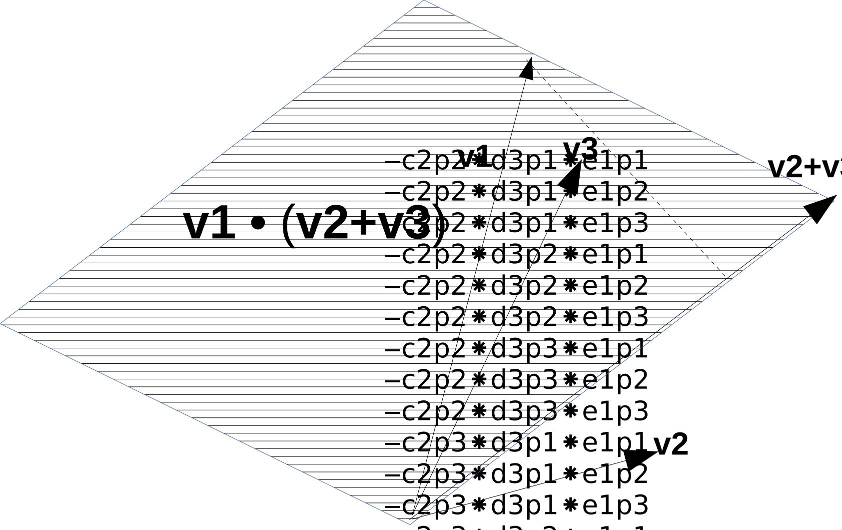
Then
x11•x23✱x12•x21✱x13•x22
= c3p1✱d3p1✱e3p1
+c3p1✱d3p1✱e3p2
+c3p1✱d3p1✱e3p3
+c3p1✱d3p2✱e3p1
+c3p1✱d3p2✱e3p2
+c3p1✱d3p2✱e3p3
+c3p1✱d3p3✱e3p1
+c3p1✱d3p3✱e3p2
+c3p1✱d3p3✱e3p3
+c3p2✱d3p1✱e3p1
+c3p2✱d3p1✱e3p2
+c3p2✱d3p1✱e3p3
+c3p2✱d3p2✱e3p1
+c3p2✱d3p2✱e3p2
+c3p2✱d3p2✱e3p3
+c3p2✱d3p3✱e3p1
+c3p2✱d3p3✱e3p2
+c3p2✱d3p3✱e3p3
+c3p3✱d3p1✱e3p1
+c3p3✱d3p1✱e3p2
+c3p3✱d3p1✱e3p3
+c3p3✱d3p2✱e3p1
+c3p3✱d3p2✱e3p2
+c3p3✱d3p2✱e3p3
+c3p3✱d3p3✱e3p1
+c3p3✱d3p3✱e3p2
+c3p3✱d3p3✱e3p3

Some the addends are negative. They expand as follows.
For
−x11•x21✱x12•x23✱x13•x22



=−(q111∗q211 + q112∗ q212 + q113∗q213**)**
∗(q121∗q231 + q122∗ q232 + q123∗q233**)**
∗(q131∗ q221 + q132∗ q222 + q133∗q223**)**
= **−**c1p1∗d2p1∗e3p1
−c1p1∗d2p1∗e3p2
−c1p1∗d2p1∗e3p3
−c1p1∗d2p2∗e3p1
−c1p1∗d2p2∗e3p2
−c1p1∗d2p2∗e3p3
−c1p1∗d2p3∗e3p1
−c1p1∗d2p3∗e3p2
−c1p1∗d2p3∗e3p3
−c1p2∗d2p1∗e3p1
−c1p2∗d2p1∗e3p2
−c1p2∗d2p1∗e3p3
−c1p2∗d2p2∗e3p1
−c1p2∗d2p2∗e3p2
−c1p2∗d2p2∗e3p3
−c1p2∗d2p3∗e3p1
−c1p2∗d2p3∗e3p2
−c1p2∗d2p3∗e3p3
−c1p3∗d2p1∗e3p1
−c1p3∗d2p1∗e3p2
−c1p3∗d2p1∗e3p3
−c1p3∗d2p2∗e3p1
−c1p3∗d2p2∗e3p2
−c1p3∗d2p2∗e3p3
−c1p3∗d2p3∗e3p1
−c1p3∗d2p3∗e3p2
−c1p3∗d2p3∗e3p3

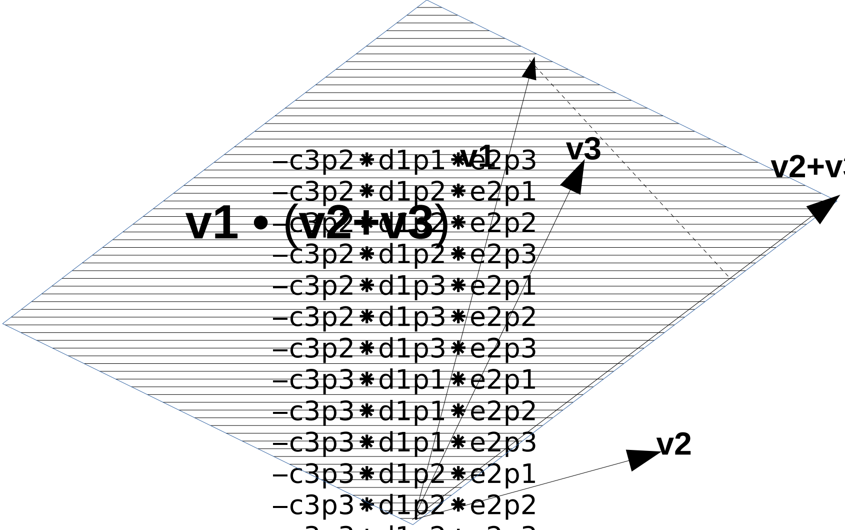
For
−x11•x22∗x12•x21)∗x13•x23
= −(q111∗q221 + q112∗q222 + q113∗q223**)**
∗(q121∗q211 + q122∗q212 + q123∗q213**)**
∗(q131∗q231 + q132∗q232 + q133∗q233**)**
= **−**c2p1∗d3p1∗e1p1
−c2p1∗d3p1∗e1p2
−c2p1∗d3p1∗e1p3
−c2p1∗d3p2∗e1p1
−c2p1∗d3p2∗e1p2
−c2p1∗d3p2∗e1p3
−c2p1∗d3p3∗e1p1
−c2p1∗d3p3∗e1p2
−c2p1∗d3p3∗e1p3



−c2p2∗d3p1∗e1p1
−c2p2∗d3p1∗e1p2
−c2p2∗d3p1∗e1p3
−c2p2∗d3p2∗e1p1
−c2p2∗d3p2∗e1p2
−c2p2∗d3p2∗e1p3
−c2p2∗d3p3∗e1p1
−c2p2∗d3p3∗e1p2
−c2p2∗d3p3∗e1p3
−c2p3∗d3p1∗e1p1
−c2p3∗d3p1∗e1p2
−c2p3∗d3p1∗e1p3
−c2p3∗d3p2∗e1p1
−c2p3∗d3p2∗e1p2
−c2p3∗d3p2∗e1p3
−c2p3∗d3p3∗e1p1
−c2p3∗d3p3∗e1p2
−c2p3∗d3p3∗e1p3

one is missing here. Corrected above.
−c2p3∗d3p1∗e1p1
−c2p3∗d3p2∗e1p1
−c2p3∗d3p1∗e1p2
−c2p3∗d3p2∗e1p2
−c2p3∗d3p3∗e1p2
−c2p3∗d3p1∗e1p3
−c2p3∗d3p2∗e1p3
−c2p3∗d3p3∗e1p3

For
− x11•x23∗x12•x22∗x13•x21
= −(q111∗q231 + q112∗q232 + q113∗q233**)**
∗(q121∗q221 + q122∗ q222 + q123∗q223**)**
∗(q131∗q211 + q132∗ q212(e1p2) + q133∗q213**)**
= **−**c3p1∗d1p1∗e2p1
−c3p1∗d1p1∗e2p2
−c3p1∗d1p1∗e2p3
−c3p1∗d1p2∗e2p1
−c3p1∗d1p2∗e2p2
−c3p1∗d1p2∗e2p3
−c3p1∗d1p3∗e2p1
−c3p1∗d1p3∗e2p2
−c3p1∗d1p3∗e2p3
−c3p2∗d1p1∗e2p1
−c3p2∗d1p1∗e2p2



–c3p2∗d1p1∗e2p3
–c3p2∗d1p2∗e2p1
–c3p2∗d1p2∗e2p2
–c3p2∗d1p2∗e2p3
–c3p2∗d1p3∗e2p1
–c3p2∗d1p3∗e2p2
–c3p2∗d1p3∗e2p3
–c3p3∗d1p1∗e2p1
–c3p3∗d1p1∗e2p2
–c3p3∗d1p1∗e2p3
–c3p3∗d1p2∗e2p1
–c3p3∗d1p2∗e2p2
–c3p3∗d1p2∗e2p3
–c3p3∗d1p3∗e2p1
–c3p3∗d1p3∗e2p2
–c3p3∗d1p3∗e2p3

Newton: "How can you put down the answers so quickly?

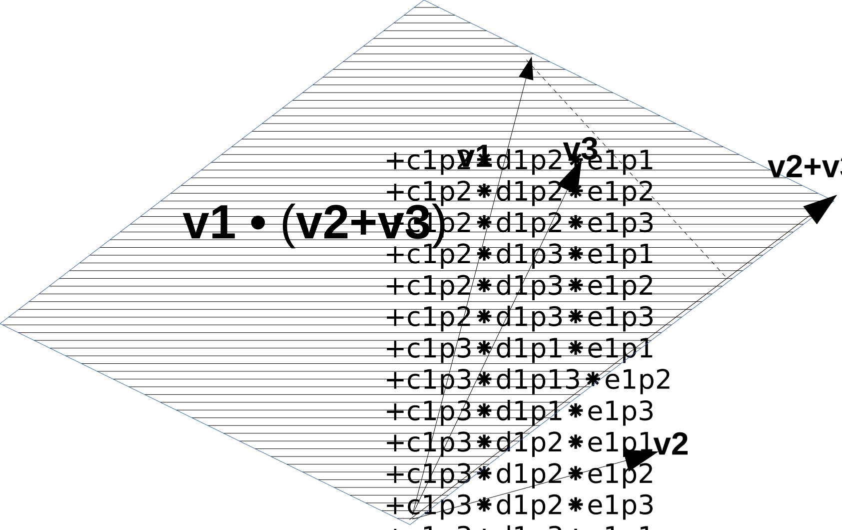
Breton: "By substituting. Once we know the expansion for **x11•x22∗x12•x23∗x13•x21** then the expansion for **x11•x21∗x12•x22∗x13•x23** simply substitutes

x21 for **x22**
x22 for **x23**
x23 for **x21**

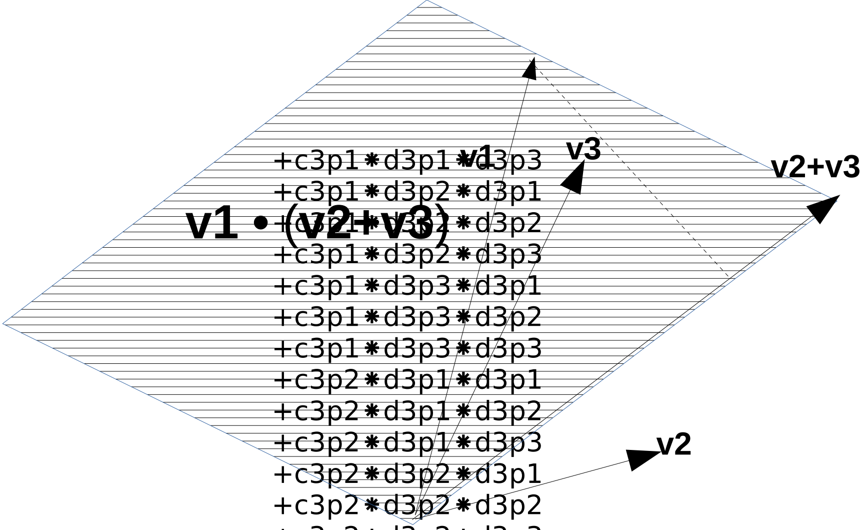
Einstein: "You still have a great many more factors for det(**X1•T[X2**]) than for det(**X1)**∗det(**X2**).

Breton: "Some may cancel. Let's look. Below I list all the positive summands followed by the negative summands.

+c1p1∗d1p1∗e1p1
+c1p1∗d1p1∗e1p2
+c1p1∗d1p1∗e1p3
+c1p1∗d1p2∗e1p1
+c1p1∗d1p2∗e1p2
+c1p1∗d1p2∗e1p3
+c1p1∗d1p3∗e1p1
+c1p1∗d1p3∗e1p2
+c1p1∗d1p3∗e1p3
+c1p2∗d1p1∗e1p1
+c1p2∗d1p1∗e1p2
+c1p2∗d1p1∗e1p3

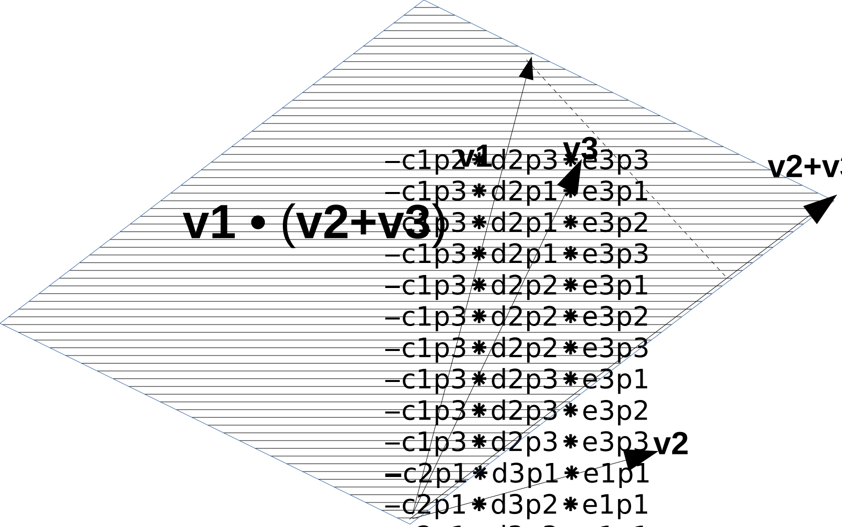


+c1p2✱d1p2✱e1p1
+c1p2✱d1p2✱e1p2
+c1p2✱d1p2✱e1p3
+c1p2✱d1p3✱e1p1
+c1p2✱d1p3✱e1p2
+c1p2✱d1p3✱e1p3
+c1p3✱d1p1✱e1p1
+c1p3✱d1p13✱e1p2
+c1p3✱d1p1✱e1p3
+c1p3✱d1p2✱e1p1
+c1p3✱d1p2✱e1p2
+c1p3✱d1p2✱e1p3
+c1p3✱d1p3✱e1p1
+c1p3✱d1p3✱e1p2
+c1p3✱d1p3✱e1p3
+c2p1✱d2p1✱e2p1 cancels −c2p1✱d3p1✱e1p1
+c2p1✱d2p1✱e2p2
+c2p1✱d2p1✱e2p3
+c2p1✱d2p2✱e2p1
+c2p1✱d2p2✱e2p2
+c2p1✱d2p2✱e2p3
+c2p1✱d2p3✱e2p1
+c2p1✱d2p3✱e2p2
+c2p1✱d2p3✱e2p3
+c2p2✱d2p1✱e2p1
+c2p2✱d2p1✱e2p2
+c2p2✱d2p1✱e2p3
+c2p2✱d2p2✱e2p1
+c2p2✱d2p2✱e2p2
+c2p2✱d2p2✱e2p3
+c2p2✱d2p3✱e2p1
+c2p2✱d2p3✱e2p2
+c2p2✱d2p3✱e2p3
+c2p3✱d2p1✱e2p1
+c2p3✱d2p1✱e2p2
+c2p3✱d2p1✱e2p3
+c2p3✱d2p2✱e2p1
+c2p3✱d2p2✱e2p2
+c2p3✱d2p2✱e2p3
+c2p3✱d2p3✱e2p1
+c2p3✱d2p3✱e2p2
+c2p3✱d2p3✱e2p3
+c3p1✱d3p1✱d3p1 cancels −c3p1✱d1p1✱e2p1
+c3p1✱d3p1✱d3p2

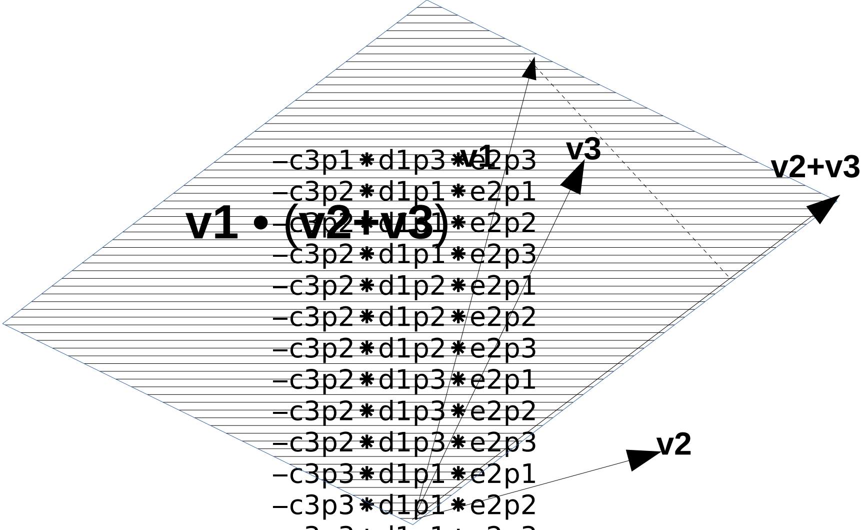


+c3p1∗d3p1∗d3p3
+c3p1∗d3p2∗d3p1
+c3p1∗d3p2∗d3p2
+c3p1∗d3p2∗d3p3
+c3p1∗d3p3∗d3p1
+c3p1∗d3p3∗d3p2
+c3p1∗d3p3∗d3p3
+c3p2∗d3p1∗d3p1
+c3p2∗d3p1∗d3p2
+c3p2∗d3p1∗d3p3
+c3p2∗d3p2∗d3p1
+c3p2∗d3p2∗d3p2
+c3p2∗d3p2∗d3p3
+c3p2∗d3p3∗d3p1
+c3p2∗d3p3∗d3p2
+c3p2∗d3p3∗d3p3
+c3p3∗d3p1∗d3p1
+c3p3∗d3p1∗d3p2
+c3p3∗d3p1∗d3p3
+c3p3∗d3p2∗d312
+c3p3∗d3p2∗d3p2
+c3p3∗d3p2∗d3p3
+c3p3∗d3p3∗d3p1
+c3p3∗d3p3∗d3p2
+c3p3∗d3p3∗d3p3

−c1p1∗d2p1∗e3p1
−c1p1∗d2p1∗e3p2
−c1p1∗d2p1∗e3p3
−c1p1∗d2p2∗e3p1
−c1p1∗d2p2∗e3p2
−c1p1∗d2p2∗e3p3
−c1p1∗d2p3∗e3p1
−c1p1∗d2p3∗e3p2
−c1p1∗d2p3∗e3p3
−c1p2∗d2p1∗e3p1
−c1p2∗d2p1∗e3p2
−c1p2∗d2p1∗e3p3
−c1p2∗d2p2∗e3p1
−c1p2∗d2p2∗e3p2
−c1p2∗d2p2∗e3p3
−c1p2∗d2p3∗e3p1
−c1p2∗d2p3∗e3p2



−c1p2∗d2p3∗e3p3
−c1p3∗d2p1∗e3p1
−c1p3∗d2p1∗e3p2
−c1p3∗d2p1∗e3p3
−c1p3∗d2p2∗e3p1
−c1p3∗d2p2∗e3p2
−c1p3∗d2p2∗e3p3
−c1p3∗d2p3∗e3p1
−c1p3∗d2p3∗e3p2
−c1p3∗d2p3∗e3p3
−c2p1∗d3p1∗e1p1
−c2p1∗d3p2∗e1p1
−c2p1∗d3p3∗e1p1
−c2p1∗d3p1∗e1p2
−c2p1∗d3p2∗e1p2
−c2p1∗d3p3∗e1p2
−c2p1∗d3p1∗e1p3
−c2p1∗d3p2∗e1p3
−c2p1∗d3p3∗e1p3
−c2p2∗d3p1∗e1p1
−c2p2∗d3p2∗e1p1
−c2p2∗d3p3∗e1p1
−c2p2∗d3p1∗e1p2
−c2p2∗d3p2∗e1p2
−c2p2∗d3p3∗e1p2
−c2p2∗d3p1∗e1p3
−c2p2∗d3p2∗e1p3
−c2p2∗d3p3∗e1p3
−c2p3∗d3p1∗e1p1
−c2p3∗d3p2∗e1p1
−c2p3∗d3p1∗e1p2
−c2p3∗d3p2∗e1p2
−c2p3∗d3p3∗e1p2
−c2p3∗d3p1∗e1p3
−c2p3∗d3p2∗e1p3
−c2p3∗d3p3∗e1p3
−c3p1∗d1p1∗e2p1
−c3p1∗d1p1∗e2p2
−c3p1∗d1p1∗e2p3
−c3p1∗d1p2∗e2p1
−c3p1∗d1p2∗e2p2
−c3p1∗d1p2∗e2p3
−c3p1∗d1p3∗e2p1
−c3p1∗d1p3∗e2p2



−c3p1∗d1p3∗e2p3
−c3p2∗d1p1∗e2p1
−c3p2∗d1p1∗e2p2
−c3p2∗d1p1∗e2p3
−c3p2∗d1p2∗e2p1
−c3p2∗d1p2∗e2p2
−c3p2∗d1p2∗e2p3
−c3p2∗d1p3∗e2p1
−c3p2∗d1p3∗e2p2
−c3p2∗d1p3∗e2p3
−c3p3∗d1p1∗e2p1
−c3p3∗d1p1∗e2p2
−c3p3∗d1p1∗e2p3
−c3p3∗d1p2∗e2p1
−c3p3∗d1p2∗e2p2
−c3p3∗d1p2∗e2p3
−c3p3∗d1p3∗e2p1
−c3p3∗d1p3∗e2p2
−c3p3∗d1p3∗e2p3

Einstein: "So none of them match! There are 27 times 6 = 162 in this cde list which cannot possibly match the 36 addends in the ab list.

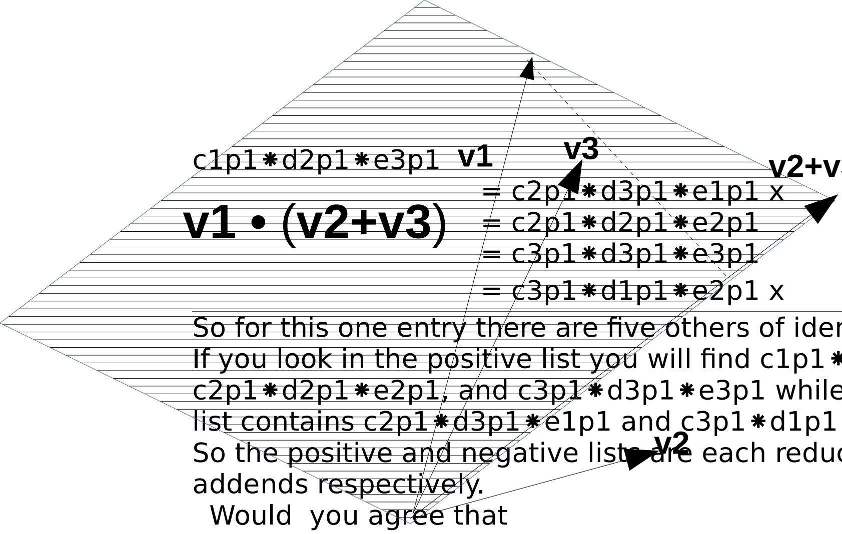
Breton: "None match formally, but perhaps in value. For instance,

c1p1∗d1p1∗e1p1 = .q111∗q211∗q121∗q221∗q131∗q231
= q111∗q211∗q121∗q231∗q131∗q221
= c1p1∗d2p1∗e3p1
= q111∗q221∗q121∗q231∗q131∗q211
= c2p1∗d2p1∗e2p1
= q111∗q221∗q121∗q211∗q131∗q231
= c2p1∗d3p1∗e1p1
= q111∗q231∗q121∗q221∗q131∗q231
= c3p1∗d1p1∗e2p1
= q111∗q231∗q121∗q211∗q131∗q221
=c3p1∗d3p1∗e3p1

So

c1p1∗d1p1∗e1p1 = c1p1∗d2p1∗e3p1
= c2p1∗d3p1∗e1p1
= c2p1∗d2p1∗e2p1
= c3p1∗d3p1∗e3p1
= c3p1∗d1p1∗e2p1

Cancellations are indicated.



c1p1*d2p1*e3p1
= c2p1*d3p1*e1p1 x
= c2p1*d2p1*e2p1
= c3p1*d3p1*e3p1
= c3p1*d1p1*e2p1 x

So for this one entry there are five others of identical value. If you look in the positive list you will find c1p1*d1p1*e1p1, c2p1*d2p1*e2p1, and c3p1*d3p1*e3p1 while the negative list contains c2p1*d3p1*e1p1 and c3p1*d1p1*e2p1. So the positive and negative lists are each reduced by two addends respectively.

Would you agree that

c1p2*d1p2*e1p2 = c1p2*d2p2*e3p2
= c2p2*d3p2*e1p2
= c2p2*d2p2*e2p2
= c3p2*d3p2*e3p2
= c3p2*d1p2*e2p2?

Einstein: "Let me heck it out. We musn't be caught up in an empty formalism.

c1p2*d1p2*e1p2 = q112*q212* q122*q222*q132*q232
while
c1p*d1p1*e1p1 = q111*q211*q121*q221*q131*q231
so the only change will be to replace final 1s with 2s.
So now let me check
c3p2*d1p2*e2p2 = q112*q232* q122*q222*q132*q212
so it works out in this instance.

Breton: "You might more easily have noted that the change from p1 to p2 simply changes qxx1 to qxx2 in all instances.

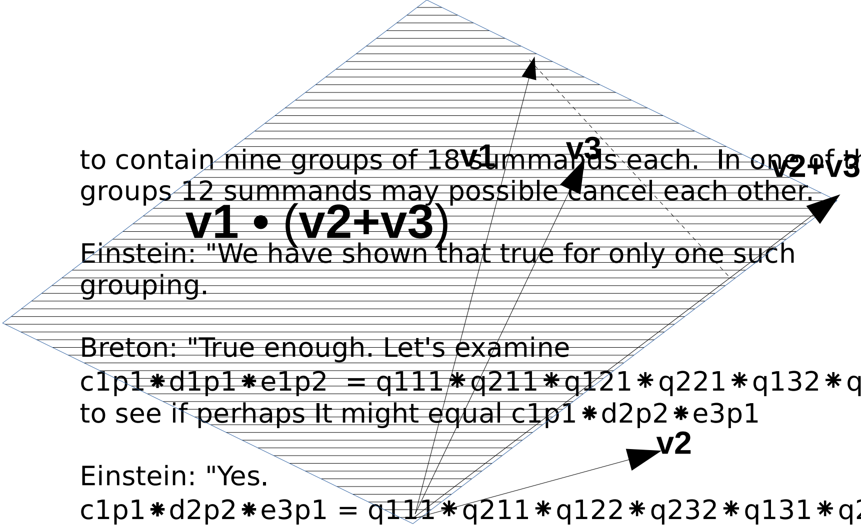
Einstein: "So I see. And changing p1 to p3 simply changes qxx1 to qxx3

Breton: "So are you ready to admit also

c1p3*d1p3*e1p3 = c1p3*d2p3*e3p3
= c2p3*d3p3*e1p3
= c2p3*d2p3*e2p3
= c3p3*d3p3*e3p3
= c3p3*d1p3*e2p3?

Einstein: "Of course.

Breton: "The 162 summands of det(**X1•X2**) can now be seen



to contain nine groups of 18 summands each. In one of these groups 12 summands may possible cancel each other.

Einstein: "We have shown that true for only one such grouping.

Breton: "True enough. Let's examine
c1p1*d1p1*e1p2 = q111*q211*q121*q221*q132*q232
to see if perhaps It might equal c1p1*d2p2*e3p1

Einstein: "Yes.
c1p1*d2p2*e3p1 = q111*q211*q122*q232*q131*q221
So these don't match, Breton.

Breton: "Let's try if c2p1*d3p1*e1p2 corresponds to
c2p1*d3p1*e1p2 = q111*q221*q121*q211*q132*q232
and so it does. It appears then that we can switch only factors with the same pi's. So with c1p1*d2p2*e3p1 we can find six corresponding summands, for c1p1*d1p1*e1p2 we can find only two.

Hh so where are the 18 summands for this situation?

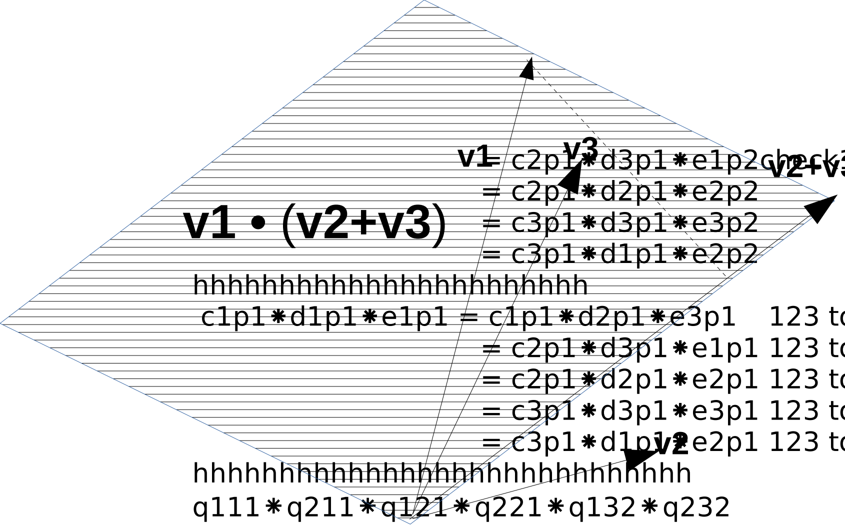
Comparisons

c1p1*d1p1*e1p1	c1p1*d1p1*e1p2
= c1p1*d2p1*e3p1 123 to 132	no
= c2p1*d3p1*e1p1 123 to 213	= c2p1*d3p1*e1p2
= c2p1*d2p1*e2p1 123 to 231	no
= c3p1*d3p1*e3p1 123 to 312	no
= c3p1*d1p1*e2p1 123 to 321	no

+c1p1*d1p2*e1p1
        = q111*q211* q122*q222* q131*q231
        = c3p1* d1p2*e2p1

+c1p1*d1p2*e1p2
        = q111*q211 q122*q222 q132*q232
        =–c1p1*d2p2*e3p2

Breton: "Then applying the same rule for substitution do you agree
   c1p1*d1p1*e1p2 = c1p1*d2p2*e3p1check 123 to 132



v1 • (v2+v3)

= c2p1*d3p1*e1p2 check 123 to 213
= c2p1*d2p1*e2p2 231
= c3p1*d3p1*e3p2 312
= c3p1*d1p1*e2p2 321

hhhhhhhhhhhhhhhhhhhhhhhh

c1p1*d1p1*e1p1 = c1p1*d2p1*e3p1 123 to 132
= c2p1*d3p1*e1p1 123 to 213
= c2p1*d2p1*e2p1 123 to 231
= c3p1*d3p1*e3p1 123 to 312
= c3p1*d1p1*e2p1 123 to 321

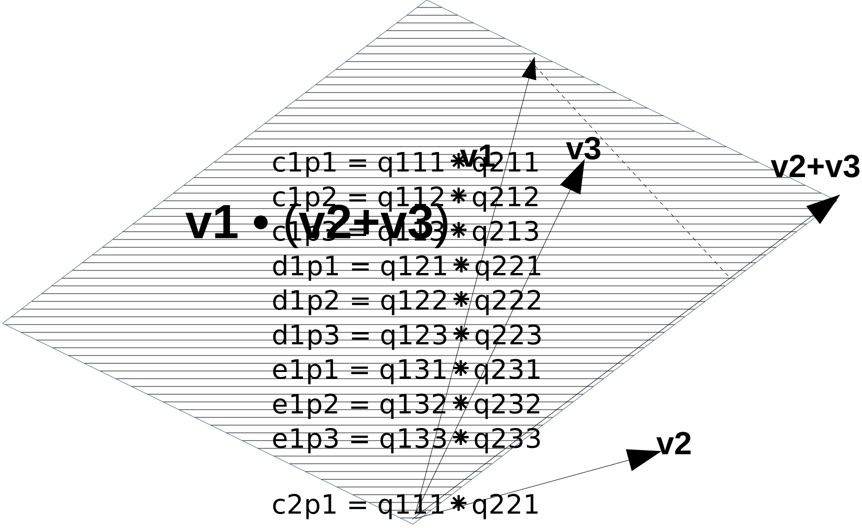
hhhhhhhhhhhhhhhhhhhhhhhhhhhhhh

q111*q211*q121*q221*q132*q232
c1p1*d1p1*e1p2
q111*q221*q121*q211*q132*q232
c2p1*d3p1*e1p2
these don't match
q111*q211*d2p2 = q122*q232*e3p1 = q131*q221
c1p1*d2p2*e3p1

Einstein: "I'll agree to c2p1*d2p1*e2p2 and
c3p1*d3p1*e3p2 but let me check
c2p1*d3p1*e1p2 = q111*q221*q121*q211*q132*q232
c2p1 = q111*q221 d3p1 = q121*q211 e1p2 = q132*q232
which checks
and
c3p1*d1p1*e2p2
c3p1 = q111*q231
d1p1 = q121*q221
e2p1 = q131*q211

.

c1p1*d1p1*e1p1 = c1p1*d2p1*e3p1
= c2p1*d3p1*e1p1
= c2p1*d2p1*e2p1
= c3p1*d3p1*e3p1
= c3p1*d1p1*e2p1



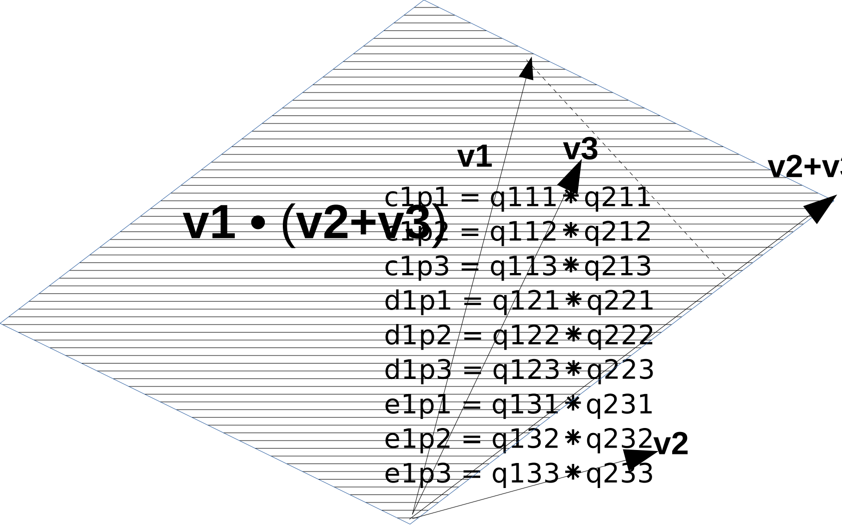
c1p1 = q111∗q211
c1p2 = q112∗q212
c1p3 = q113∗q213
d1p1 = q121∗q221
d1p2 = q122∗q222
d1p3 = q123∗q223
e1p1 = q131∗q231
e1p2 = q132∗q232
e1p3 = q133∗q233

c2p1 = q111∗q221
c2p2 = q112∗q222
c2p3 = q113∗q223
d2p1 = q121∗q231
d2p2 = q122∗q232
d2p3 = q123∗q233
e2p1 = q131∗q211
e2p2 = q132∗q212
e2p3 = q133∗q213

c3p1 = q111∗q231
c3p2 = q112∗q232
c3p3 = q113∗q233
d3p1 = q121∗q211
d3p2 = q122∗q212
d3p3 = q123∗q213
e3p1 = q131∗q221
e3p2 = q132∗q222
e3p3 = q133∗q223

Breton: "
Now we need to see how the labels fit together.
ap1∗bp1= q112∗q123∗q131∗q212∗q223∗q231
= c3p2∗q123∗q131∗q223∗q231 (c3p2 =q112∗q212)
= c3p2∗d3p3∗q131∗q231 (d3p3 =q123∗q223)
= c3p2∗d3p3∗e3p1 (e3p1 =q131∗q231)
so

$$\frac{ap1*bp1=c1p2*d1p3*e1p1}{q111*q122*q133}$$



c1p1 = q111*q211
c1p2 = q112*q212
c1p3 = q113*q213
d1p1 = q121*q221
d1p2 = q122*q222
d1p3 = q123*q223
e1p1 = q131*q231
e1p2 = q132*q232
e1p3 = q133*q233

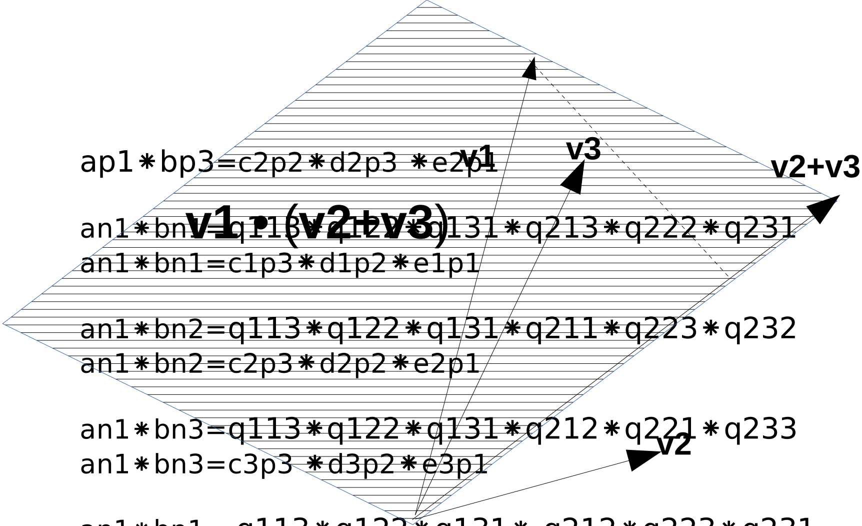
c2p1 = q111*q221
c2p2 = q112*q222
c2p3 = q113*q223
d2p1 = q121*q231
d2p2 = q122*q232
d2p3 = q123*q233
e2p1 = q131*q211
e2p2 = q132*q212
e2p3 = q133*q213

c3p1 = q111*q231
c3p2 = q112*q232
c3p3 = q113*q233
d3p1 = q121*q211
d3p2 = q122*q212
d3p3 = q123*q213
e3p1 = q131*q221
e3p2 = q132*q222
e3p3 = q133*q223

ap1*bp1=q112*q123*q131*q212*q223*q231
ap1*bp1=c1p2*d1p3*e1p1

ap1*bp2 =q112*q123*q131*q213*q221*q232
ap1*bp2 =c3p2 *d3p3 *e3p1

ap1*bp3= q112*q123*q131*q211*q222*q233



ap1∗bp3=c2p2∗d2p3 ∗e2p1

an1∗bn1=q113∗q122∗q131∗q213∗q222∗q231
an1∗bn1=c1p3∗d1p2∗e1p1

an1∗bn2=q113∗q122∗q131∗q211∗q223∗q232
an1∗bn2=c2p3∗d2p2∗e2p1

an1∗bn3=q113∗q122∗q131∗q212∗q221∗q233
an1∗bn3=c3p3 ∗d3p2∗e3p1

an1∗bp1= q113∗q122∗q131∗ q212∗q223∗q231
an1∗bp1 = c2p3 ∗d3p2∗e1p1

an1∗bp2= q113∗q122∗q131∗q213∗q221∗q232
an1∗bp2= c1p3∗d2p2 ∗e3p1

an1∗bp3= q113∗q122∗q131∗ q211∗q222∗q233
an1∗bp3= c3p3 ∗d1p2 ∗e2p1

ap1∗bn1= q112∗q123∗q131∗q213∗q222∗q231
ap1∗bn1= c2p2 ∗d3p3 ∗e1p1

ap1∗bn2= q112∗q123∗q131∗q211∗q223∗q232
ap1∗bn2= c3p2∗d1p3∗e2p1

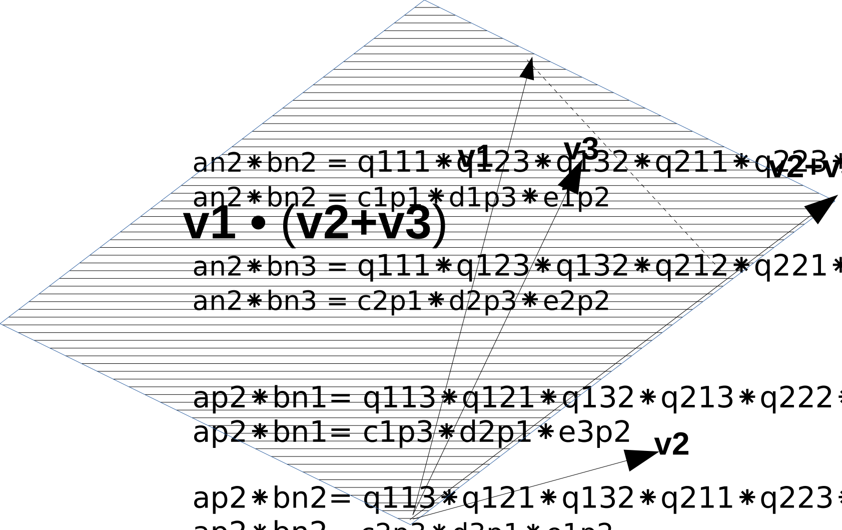
ap1∗bn3= q112∗q123∗q131∗q212∗q221∗q233
ap1∗bn3= c1p2 ∗d2p3 ∗e3p1

ap2∗bp1= q113∗q121∗q132∗q212∗q223∗q231
ap2∗bp1= c2p3∗d2p1∗e2p2

ap2∗bp2= q113∗q121∗q132∗q213∗q221∗q232
ap2∗bp2 =c1p3∗d1p1∗e1p2

ap2∗bp3= q113∗q121∗q132∗q211∗q222∗q233
ap2∗bp3=c3p3∗d3p1∗e3p2

an2∗bn1 = q111∗q123∗q132∗q213∗q222∗q231
an2∗bn1 = c3p1∗d3p3∗e3p2



an2∗bn2 = q111∗q123∗q132∗q211∗q223∗q232
an2∗bn2 = c1p1∗d1p3∗e1p2

v1 • (v2+v3)

an2∗bn3 = q111∗q123∗q132∗q212∗q221∗q233
an2∗bn3 = c2p1∗d2p3∗e2p2

ap2∗bn1= q113∗q121∗q132∗q213∗q222∗q231
ap2∗bn1= c1p3∗d2p1∗e3p2

ap2∗bn2= q113∗q121∗q132∗q211∗q223∗q232
ap2∗bn2 =c2p3∗d3p1∗e1p2

ap2∗bn3= q113∗q121∗q132∗q212∗q221∗q233
ap2∗bn3=c3p3∗d1p1∗e2p2

an2∗bp1 = q111∗q123∗q132∗q212∗q223∗q231
an2∗bp1 = c3p1∗d1p3∗e2p2

an2∗bp2 = q111∗q123∗q132∗q213∗q221∗q232
an2∗bp2 = c2p1∗d3p3∗e1p2

an2∗bp3 = q111∗q123∗q132∗q211∗q222∗q233
an2∗bp3 = c1p1∗d2p3∗e3p2

ap3∗bp1=q111∗q122∗q133∗q212∗q223∗q231
ap3∗bp1=c3p1∗d3p2∗e3p3

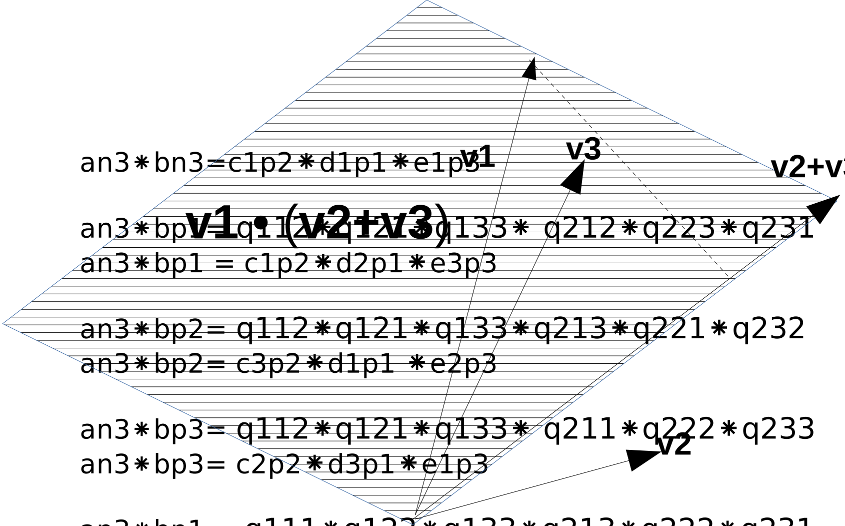
ap3∗bp2 =q111∗q122∗q133∗q213∗q221∗q232
ap3∗bp2 =c2p1 ∗d2p2 ∗e2p3

ap3∗bp3= q111∗q122∗q133∗q211∗q222∗q233
ap3∗bp3=c1p1∗d1p2 ∗e1p3

an3∗bn1=q112∗q121∗q133∗q213∗q222∗q231
an3∗bn1=c2p2∗d2p1∗e2p3

an3∗bn2=q112∗q121∗q133∗q211∗q223∗q232
an3∗bn2=c3p2∗d3p1∗e3p3

an3∗bn3=q112∗q121∗q133∗q212∗q221∗q233



an3∗bn3=c1p2∗d1p1∗e1p3

an3∗bp1= q112∗q121∗q133∗ q212∗q223∗q231
an3∗bp1 = c1p2∗d2p1∗e3p3

an3∗bp2= q112∗q121∗q133∗q213∗q221∗q232
an3∗bp2= c3p2∗d1p1 ∗e2p3

an3∗bp3= q112∗q121∗q133∗ q211∗q222∗q233
an3∗bp3= c2p2∗d3p1∗e1p3

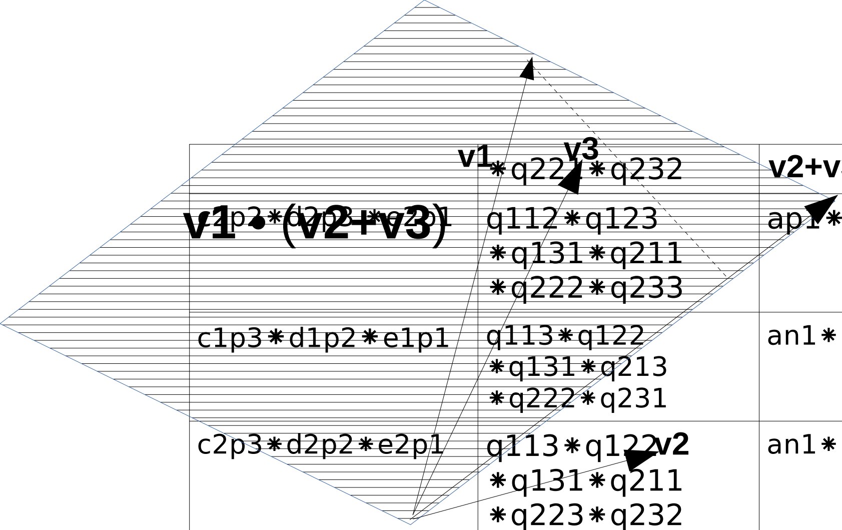
ap3∗bn1= q111∗q122∗q133∗q213∗q222∗q231
ap3∗bn1= c3p1∗d1p2∗e2p3

ap3∗bn2= q111∗q122∗q133∗q211∗q223∗q232
ap3∗bn2= c1p1∗d2p2∗e3p3

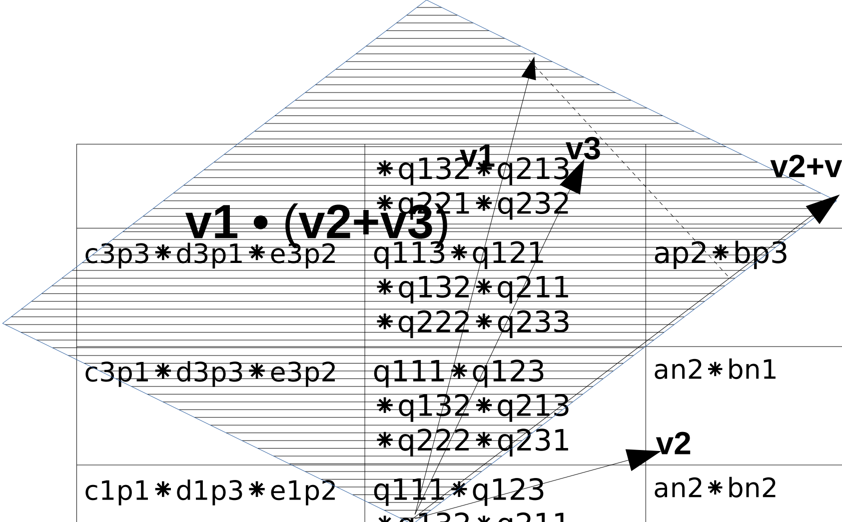
ap3∗bn3= q111∗q122∗q133∗q212∗q221∗q233
ap3∗bn3= c2p1∗d3p2∗e1p3

ap1= q112∗q123∗q131
an1= q113∗q122∗q131
ap2= q113∗q121∗q132
an2= q111∗q123∗q132
ap3= q111∗q122∗q133
an3= q112∗q121∗q133
bp1= q212∗q223∗q231
bn1= q213∗q222∗q231
bp2= q213∗q221∗q232
bn2= q211∗q223∗q232
bp3= q211∗q222∗q233
bn3= q212∗q221∗q233

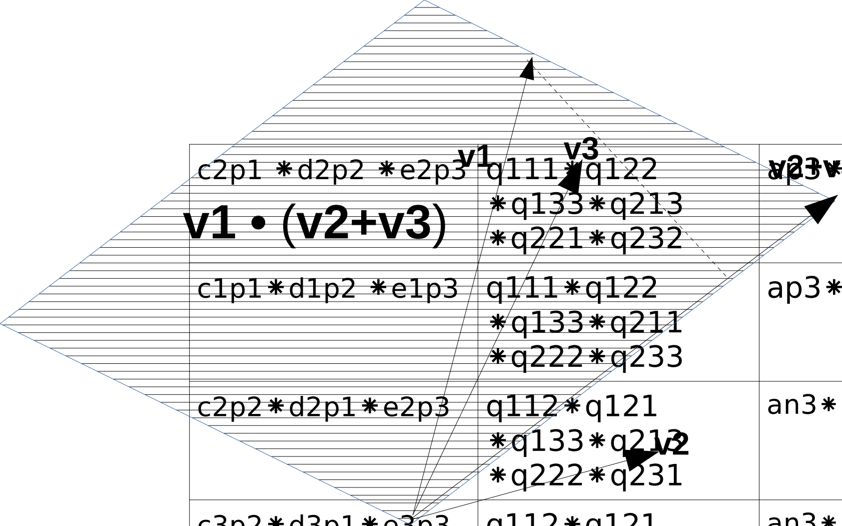
det(**X1**•**T**[**X2**])	value	det(**X1)**∗det(**X2**)
c1p2∗d1p3∗e1p1	q112∗q123 ∗q131∗q212 ∗q223∗q231	ap1∗bp1
c3p2∗d3p3 ∗e3p1	q112∗q123 ∗q131∗q213	ap1∗bp2



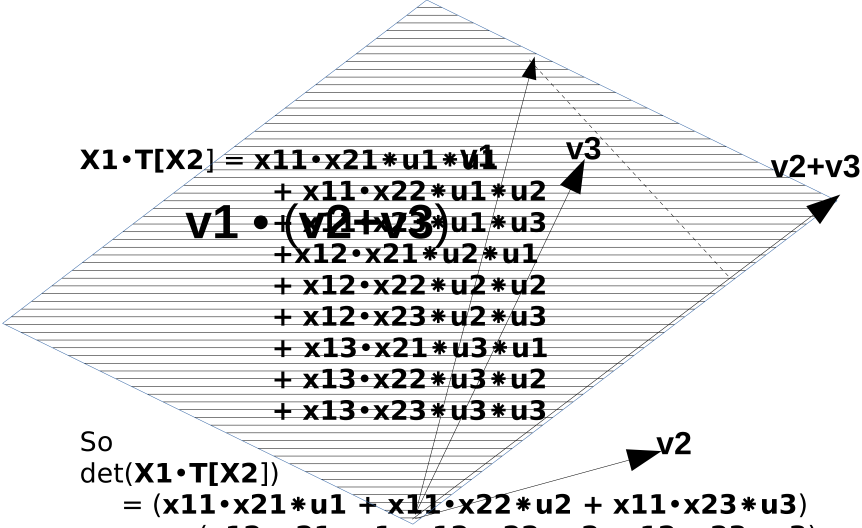
	*q221*q232	
c1p2*d2p3*e2p1	q112*q123 *q131*q211 *q222*q233	ap1*bp3
c1p3*d1p2*e1p1	q113*q122 *q131*q213 *q222*q231	an1*bn1
c2p3*d2p2*e2p1	q113*q122 *q131*q211 *q223*q232	an1*bn2
c3p3 *d3p2*e3p1	q113*q122 *q131*q212 *q221*q233	an1*bn3
c2p3 *d3p2*e1p1	q113*q122 *q131* q212 *q223*q231	−an1*bp1
c1p3*d2p2 *e3p1	q113*q122 *q131*q213 *q221*q232	−an1*bp2
c3p3 *d1p2 *e2p1	q113*q122 *q131* q211 *q222*q233	−an1*bp3
c2p2 *d3p3 *e1p1	q112*q123 *q131*q213 *q222*q231	−ap1*bn1
c3p2*d1p3*e2p1	q112*q123 *q131*q211 *q223*q232	−ap1*bn2
c1p2 *d2p3 *e3p1	q112*q123 *q131*q212 *q221*q233	−ap1*bn3
c2p3*d2p1*e2p2	q113*q121 *q132*q212 *q223*q231	ap2*bp1
c1p3*d1p1*e1p2	q113*q121	ap2*bp2



	∗q132∗q213 ∗q221∗q232	
c3p3∗d3p1∗e3p2	q113∗q121 ∗q132∗q211 ∗q222∗q233	ap2∗bp3
c3p1∗d3p3∗e3p2	q111∗q123 ∗q132∗q213 ∗q222∗q231	an2∗bn1
c1p1∗d1p3∗e1p2	q111∗q123 ∗q132∗q211 ∗q223∗q232	an2∗bn2
c2p1∗d2p3∗e2p2	q111∗q123 ∗q132∗q212 ∗q221∗q233	an2∗bn3
c1p3∗d2p1∗e3p2	q113∗q121 ∗q132∗q213 ∗q222∗q231	−ap2∗bn1
c2p3∗d3p1∗e1p2	q113∗q121 ∗q132∗q211 ∗q223∗q232	−ap2∗bn2
c3p3∗d1p1∗e2p2	q113∗q121 ∗q132∗q212 ∗q221∗q233	−ap2∗bn3
c3p1∗d1p3∗e2p2	q111∗q123 ∗q132∗q212 ∗q223∗q231	−an2∗bp1
c2p1∗d3p3∗e1p2	q111∗q123 ∗q132∗q213 ∗q221∗q232	−an2∗bp2
c1p1∗d2p3∗e3p2	q111∗q123 ∗q132∗q211 ∗q222∗q233	−an2∗bp3
c3p1∗d3p2∗e3p3	q111∗q122 ∗q133∗q212 ∗q223∗q231	ap3bp1



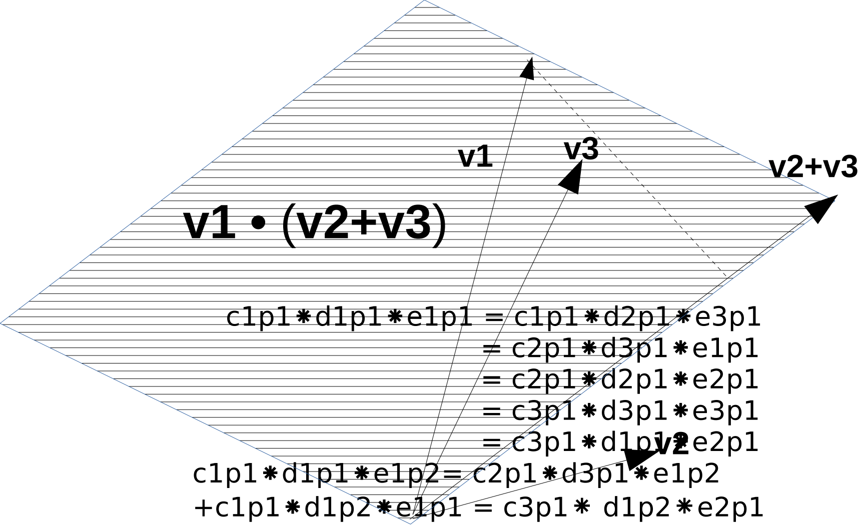
c2p1 ∗d2p2 ∗e2p3	q111∗q122 ∗q133∗q213 ∗q221∗q232	ap3∗bp2
c1p1∗d1p2 ∗e1p3	q111∗q122 ∗q133∗q211 ∗q222∗q233	ap3∗bp3
c2p2∗d2p1∗e2p3	q112∗q121 ∗q133∗q213 ∗q222∗q231	an3∗bn1
c3p2∗d3p1∗e3p3	q112∗q121 ∗q133∗q211 ∗q223∗q232	an3∗bn2
c1p2∗d1p1∗e1p3	q112∗q121 ∗q133∗q212 ∗q221∗q233	an3∗bn3
c1p2∗d2p1∗e3p3	q112∗q121 ∗q133∗q212 ∗q223∗q231	−an3∗bp1
c3p2∗d1p1∗e2p3	q112∗q121 ∗q133∗q213 ∗q221∗q232	−an3∗bp2
c2p2∗d3p1∗e1p3	q112∗q121 ∗q133∗q211 ∗q222∗q233	−an3∗bp3
c3p1∗d1p2∗e2p3	q111∗q122 ∗q133∗q213 ∗q222∗q231	−ap3∗bn1
c1p1∗d2p2∗e3p3	q111∗q122 ∗q133∗q211 ∗q223∗q232	−ap3∗bn2
c2p1∗d3p2∗e1p3	q111∗q122 ∗q133∗q212 ∗q221∗q233	−ap3∗bn3



X1•T[X2] = **x11•x21∗u1∗u1**
+ x11•x22∗u1∗u2
+ x11•x23∗u1∗u3
+x12•x21∗u2∗u1
+ x12•x22∗u2∗u2
+ x12•x23∗u2∗u3
+ x13•x21∗u3∗u1
+ x13•x22∗u3∗u2
+ x13•x23∗u3∗u3

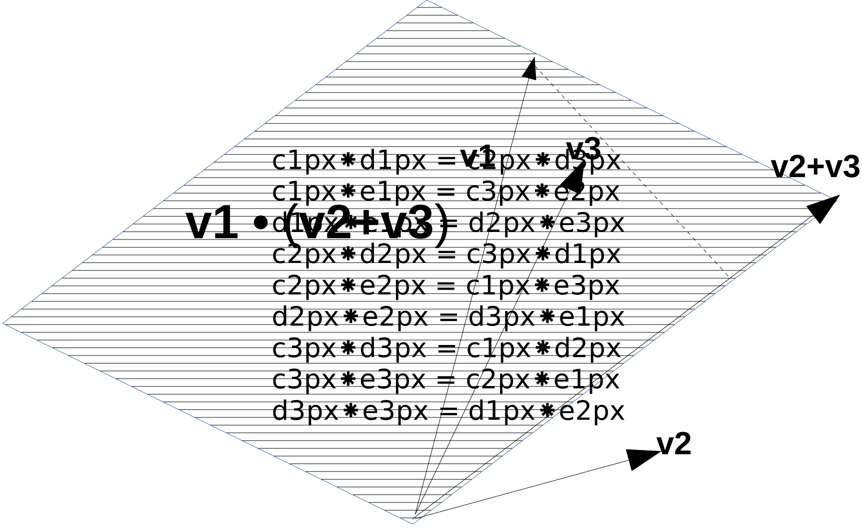
So
det(**X1•T[X2**])
= (**x11•x21∗u1 + x11•x22∗u2 + x11•x23∗u3**)
∧(**x12•x21∗u1+x12•x22∗u2+x12•x23∗u3**)
•(x13•x21∗u1+x13•x22∗u2+x13•x23∗u3)
=(**x11•x22∗x12•x23− x11•x23∗x12•x22)∗u1**
+(x11•x23∗x12•x21−x11•x21∗x12•x23)**∗u2**
+(**x11•x21∗x12•x22−x11•x22∗x12•x21)∗u3**)
•(x13•x21∗u1+x13•x22∗u2+x13•x23∗u3)
=**x11•x22∗x12•x23∗x13•x21**
− x11•x23∗x12•x22∗x13•x21
+**x11•x23∗x12•x21∗x13•x22**
−x11•x21∗x12•x23∗x13•x22
+**x11•x21∗x12•x22∗x13•x23**
−x11•x22∗x12•x21∗x13•x23
hh this all checks out.

c1p1∗d1p1∗e1p1 = c1p1∗d2p1∗e3p1
= c2p1∗d3p1∗e1p1
= c2p1∗d2p1∗e2p1
= c3p1∗d3p1∗e3p1
= c3p1∗d1p1∗e2p1
c1p2∗d1p2∗e1p2 = c1p2∗d2p2∗e3p2
= c2p2∗d3p2∗e1p2
= c2p2∗d2p2∗e2p2
= c3p2∗d3p2∗e3p2
= c3p2∗d1p2∗e2p2?
c1p3∗d1p3∗e1p3 = c1p3∗d2p3∗e3p3
= c2p3∗d3p3∗e1p3
= c2p3∗d2p3∗e2p3
= c3p3∗d3p3∗e3p3
= c3p3∗d1p3∗e2p3?

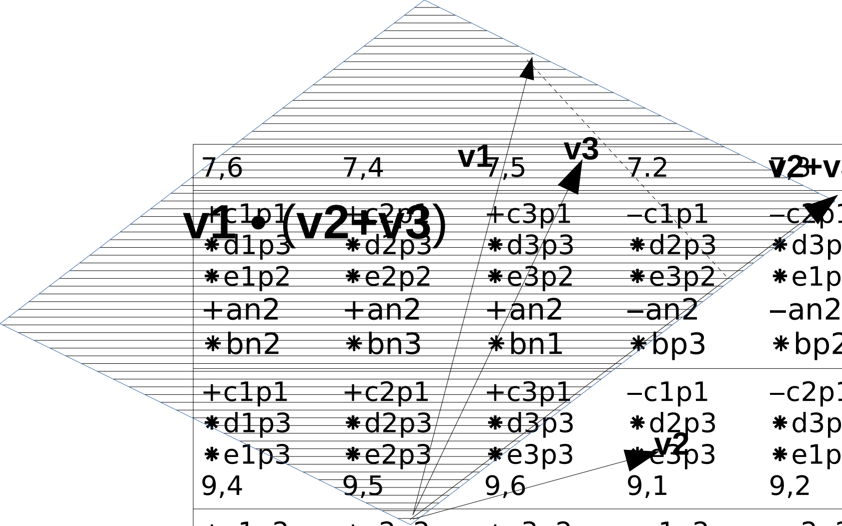


c1p1⁕d1p1⁕e1p1 = c1p1⁕d2p1⁕e3p1
= c2p1⁕d3p1⁕e1p1
= c2p1⁕d2p1⁕e2p1
= c3p1⁕d3p1⁕e3p1
= c3p1⁕d1p1⁕e2p1

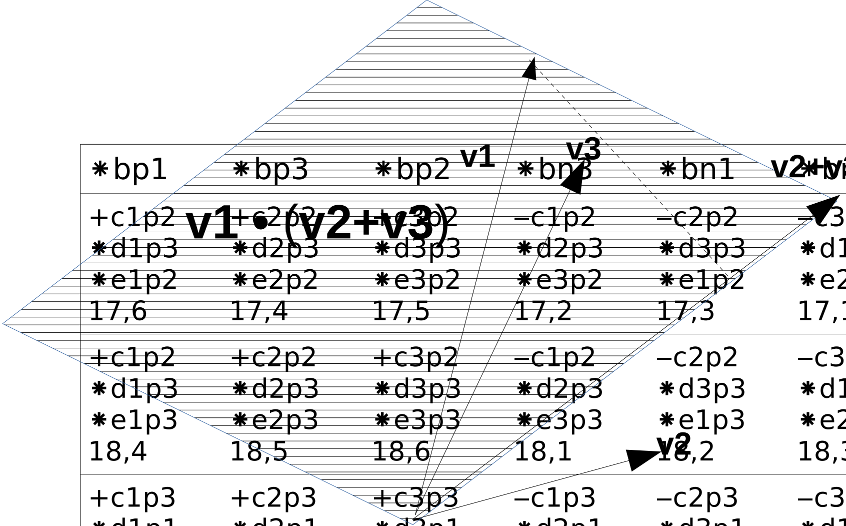
c1p1⁕d1p1⁕e1p2= c2p1⁕d3p1⁕e1p2
+c1p1⁕d1p2⁕e1p1 = c3p1⁕ d1p2⁕e2p1
+c1p1⁕d1p2⁕e1p2
= q111⁕q211 q122⁕q222 q132⁕q232
=−c1p1⁕d2p2⁕e3p2



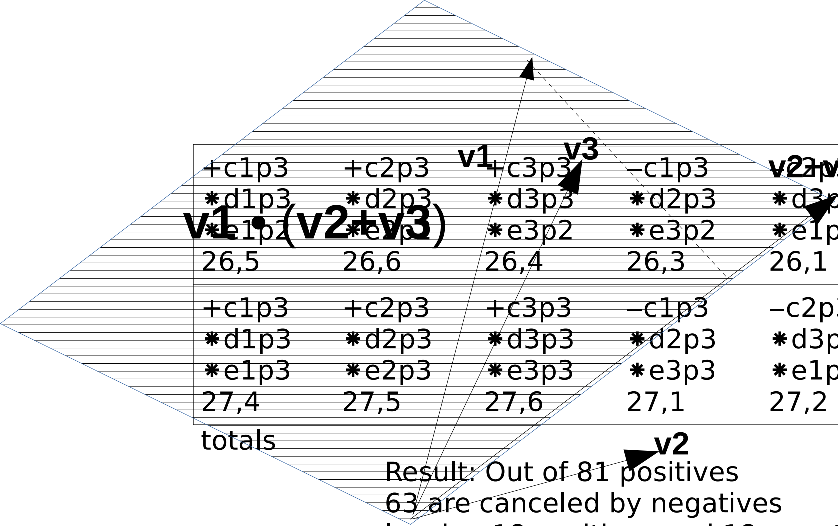
c1p1 ✱d1p1 ✱e1p1 1,4	+c2p1 ✱d2p1 ✱e2p1 1,5	+c3p1 ✱d3p1 ✱e3p1 1,6	−c1p1 ✱d2p1 ✱e3p1 1,1	−c2p1 ✱d3p1 ✱e1p1 1,2	−c3p1 ✱d1p1 ✱e2p1 1,3
+c1p1 ✱d1p1 ✱e1p2 2,5	+c2p1 ✱d2p1 ✱e2p2 2,6	+c3p1 ✱d3p1 ✱e3p2 2,4	−c1p1 ✱d2p1 ✱e3p2 2,3	−c2p1 ✱d3p1 ✱e1p2 :2 ,1	−c3p1 ✱d1p1 ✱e2p2 2,2
+c1p1 ✱d1p1 ✱e1p3 3,5	+c2p1 ✱d2p1 ✱e2p3 3,6	+c3p1 ✱d3p1 ✱e3p3 3,4	−c1p1 ✱d2p1 ✱e3p3 3,3	−c2p1 ✱d3p1 ✱e1p3 3,1	−c3p1 ✱d1p1 ✱e2p3 3,2
+c1p1 ✱d1p2 ✱e1p1 4,6	+c2p1 ✱d2p2 ✱e2p1 4,4	+c3p1 ✱d3p2 ✱e3p1 4,5	−c1p1 ✱d2p2 ✱e3p1 4,2	−c2p1 ✱d3p2 ✱e1p1 4,3	−c3p1 ✱d1p2 ✱e2p1 4,1
+c1p1 ✱d1p2 ✱e1p2 5,4	+c2p1 ✱d2p2 ✱e2p2 5,5	+c3p1 ✱d3p2 ✱e3p2 5,6	−c1p1 ✱d2p2 ✱e3p2 5,1	−c2p1 ✱d3p2 ✱e1p2 5,2	−c3p1 ✱d1p2 ✱e2p2 5,3
+c1p1 ✱d1p2 ✱e1p3 +ap3 ✱bp3	+c2p1 ✱d2p2 ✱e2p3 +ap3 ✱bp2	+c3p1 ✱d3p2 ✱e3p3 +ap3 ✱bp1	−c1p1 ✱d2p2 ✱e3p3 –ap3 ✱bn2	−c2p1 ✱d3p2 ✱e1p3 –ap3 ✱bn3	−c3p1 ✱d1p2 ✱e2p3 –ap3 ✱bn1
+c1p1 ✱d1p3 ✱e1p1	+c2p1 ✱d2p3 ✱e2p1	+c3p1 ✱d3p3 ✱e3p1	−c1p1 ✱d2p3 ✱e3p1	−c2p1 ✱d3p3 ✱e1p1	−c3p1 ✱d1p3 ✱e2p1



7,6	7,4	7,5	7.2	7,3	7,1
+c1p1 ✱d1p3 ✱e1p2 +an2 ✱bn2	+c2p1 ✱d2p3 ✱e2p2 +an2 ✱bn3	+c3p1 ✱d3p3 ✱e3p2 +an2 ✱bn1	−c1p1 ✱d2p3 ✱e3p2 −an2 ✱bp3	−c2p1 ✱d3p3 ✱e1p2 −an2 ✱bp2	−c3p1 ✱d1p3 ✱e2p2 −an2 ✱bp1
+c1p1 ✱d1p3 ✱e1p3 9,4	+c2p1 ✱d2p3 ✱e2p3 9,5	+c3p1 ✱d3p3 ✱e3p3 9,6	−c1p1 ✱d2p3 ✱e3p3 9,1	−c2p1 ✱d3p3 ✱e1p3 9,2	−c3p1 ✱d1p3 ✱e2p3 9,3
+c1p2 ✱d1p1 ✱e1p1 10,4	+c2p2 ✱d2p1 ✱e2p1 10,5	+c3p2 ✱d3p1 ✱e3p1 10,6	−c1p2 ✱d2p1 ✱e3p1 10,1	−c2p2 ✱d3p1 ✱e1p1 10,2	−c3p2 ✱d1p1 ✱e2p1 10,3
+c1p2 ✱d1p1 ✱e1p2 11,6	+c2p2 ✱d2p1 ✱e2p2 11,4	+c3p2 ✱d3p1 ✱e3p2 11,5	−c1p2 ✱d2p1 ✱e3p2 11,2	−c2p2 ✱d3p1 ✱e1p2c1 11,3	−c3p2 ✱d1p1 ✱e2p2 11,1
+c1p2 ✱d1p1 ✱e1p3 +an3 ✱bn3	+c2p2 ✱d2p1 ✱e2p3 +an3 ✱bn1	+c3p2 ✱d3p1 ✱e3p3 +an3 ✱bn2	−c1p2 ✱d2p1 ✱e3p3 −an3 ✱bp1	−c2p2 ✱d3p1 ✱e1p3 –an3 ✱bp3	−c3p2 ✱d1p1 ✱e2p3 –an3 ✱bp2
+c1p2 ✱d1p2 ✱e1p1 13,5	+c2p2 ✱d2p2 ✱e2p1 13,6	+c3p2 ✱d3p2 ✱e3p1 13,4	−c1p2 ✱d2p2 ✱e3p1 13,3	−c2p2 ✱d3p2 ✱e1p1 13,1	−c3p2 ✱d1p2 ✱e2p1 13,2
+c1p2 ✱d1p2 ✱e1p2 14,5	+c2p2 ✱d2p2 ✱e2p2 14,6	+c3p2 ✱d3p2 ✱e3p2 14,4	−c1p2 ✱d2p2 ✱e3p2 14,3	−c2p2 ✱d3p2 ✱e1p2 14,1	−c3p2 ✱d1p2 ✱e2p2 14,2
+c1p2 ✱d1p2 ✱e1p3 15,5	+c2p2 ✱d2p2 ✱e2p3 15,6	+c3p2 ✱d3p2 ✱e3p3 15,4	−c1p2 ✱d2p2 ✱e3p3 5,3	−c2p2 ✱d3p2 ✱e1p3 15,1	−c3p2 ✱d1p2 ✱e2p3 15,2
+c1p2 ✱d1p3 ✱e1p1 +ap1	+c2p2 ✱d2p3 ✱e2p1 +ap1	+c3p2 ✱d3p3 ✱e3p1 +ap1	−c1p2 ✱d2p3 ✱e3p1 –ap1	−c2p2 ✱d3p3 ✱e1p1 −ap1	−c3p2 ✱d1p3 ✱e2p1 –ap1



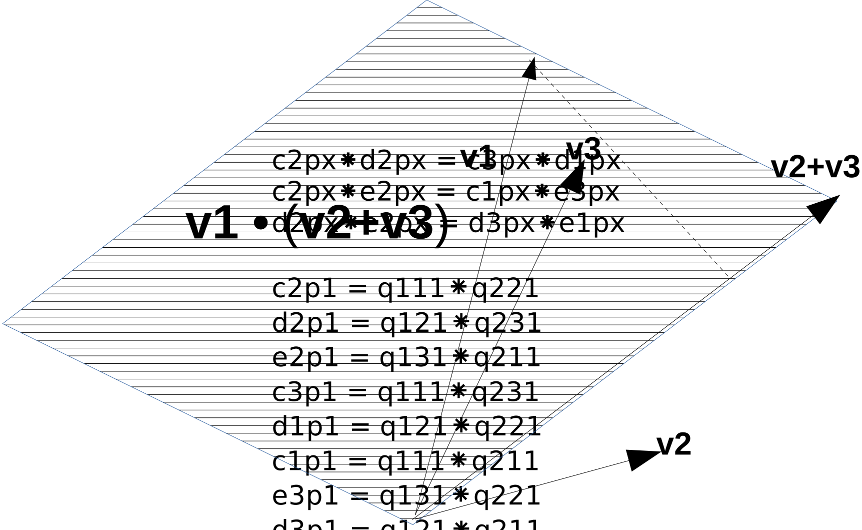
∗bp1	∗bp3	∗bp2	∗bn3	∗bn1	∗bn2
+c1p2 ∗d1p3 ∗e1p2 17,6	+c2p2 ∗d2p3 ∗e2p2 17,4	+c3p2 ∗d3p3 ∗e3p2 17,5	−c1p2 ∗d2p3 ∗e3p2 17,2	−c2p2 ∗d3p3 ∗e1p2 17,3	−c3p2 ∗d1p3 ∗e2p2 17,1
+c1p2 ∗d1p3 ∗e1p3 18,4	+c2p2 ∗d2p3 ∗e2p3 18,5	+c3p2 ∗d3p3 ∗e3p3 18,6	−c1p2 ∗d2p3 ∗e3p3 18,1	−c2p2 ∗d3p3 ∗e1p3 18,2	−c3p2 ∗d1p3 ∗e2p3 18,3
+c1p3 ∗d1p1 ∗e1p1 19,4	+c2p3 ∗d2p1 ∗e2p1 19,5	+c3p3 ∗d3p1 ∗e3p1 19,6	−c1p3 ∗d2p1 ∗e3p1 19,1	−c2p3 ∗d3p1 ∗e1p1 19,2	−c3p3 ∗d1p1 ∗e2p1 19,6
+c1p3 ∗d1p1 ∗e1p2 +ap2 ∗bp2	+c2p3 ∗d2p1 ∗e2p2 +ap2 ∗bp1	+c3p3 ∗d3p1 ∗e3p2 +ap2 ∗bp3	−c1p3 ∗d2p1 ∗e3p2 –ap2 ∗bn1	−c2p3 ∗d3p1 ∗e1p2 –ap2 ∗bn2	−c3p3 ∗d1p1 ∗e2p2 −ap2 ∗bn3
+c1p3 ∗d1p1 ∗e1p3 21,4	+c2p3 ∗d2p1 ∗e2p3 21,5	+c3p3 ∗d3p1 ∗e3p3 25,6	−c1p3 ∗d2p1 ∗e3p3 21,1	−c2p3 ∗d3p1 ∗e1p3 21,2	−c3p3 ∗d1p1 ∗e2p3 21,3
+c1p3 ∗d1p2 ∗e1p1 +an1 ∗bn1	+c2p3 ∗d2p2 ∗e2p1 +an1 ∗bn2	+c3p3 ∗d3p2 ∗e3p1 +an1 ∗bn3	−c1p3 ∗d2p2 ∗e3p1 −an1 ∗bp2	−c2p3 ∗d3p2 ∗e1p1 −an1 ∗bp1	−c3p3 ∗d1p2 ∗e2p1 −an1 ∗bp3
+c1p3 ∗d1p2 ∗e1p2 23,4	+c2p3 ∗d2p2 ∗e2p2 23,5	+c3p3 ∗d3p2 ∗e3p2 23,6	−c1p3 ∗d2p2 ∗e3p2 23,1	−c2p3 ∗d3p2 ∗e1p2 23,2	−c3p3 ∗d1p2 ∗e2p2 23,3
+c1p3 ∗d1p2 ∗e1p3 24,4	+c2p3 ∗d2p2 ∗e2p3 24,5	+c3p3 ∗d3p2 ∗e3p3 24,6	−c1p3 ∗d2p2 ∗e3p3 24,1	−c2p3 ∗d3p2 ∗e1p3 24,2	−c3p3 ∗d1p2 ∗e2p3 24,3
+c1p3 ∗d1p3 ∗e1p1 25,5	+c2p3 ∗d2p3 ∗e2p1 25,6	+c3p3 ∗d3p3 ∗e3p1 25,4	−c1p3 ∗d2p3 ∗e3p1 25,3	−c2p3 ∗d3p3 ∗e1p1 25,1	−c3p3 ∗d1p3 ∗e2p1 25,2



+c1p3 ⁕d1p3 ⁕e1p2 26,5	+c2p3 ⁕d2p3 ⁕e2p2 26,6	+c3p3 ⁕d3p3 ⁕e3p2 26,4	−c1p3 ⁕d2p3 ⁕e3p2 26,3	−c2p3 ⁕d3p3 ⁕e1p2 26,1	−c3p3 ⁕d1p3 ⁕e2p2 26,2
+c1p3 ⁕d1p3 ⁕e1p3 27,4	+c2p3 ⁕d2p3 ⁕e2p3 27,5	+c3p3 ⁕d3p3 ⁕e3p3 27,6	−c1p3 ⁕d2p3 ⁕e3p3 27,1	−c2p3 ⁕d3p3 ⁕e1p3 27,2	−c3p3 ⁕d1p3 ⁕e2p3 27,3

totals

Result: Out of 81 positives
63 are canceled by negatives
leaving 18 positives and 18 negatives addends..

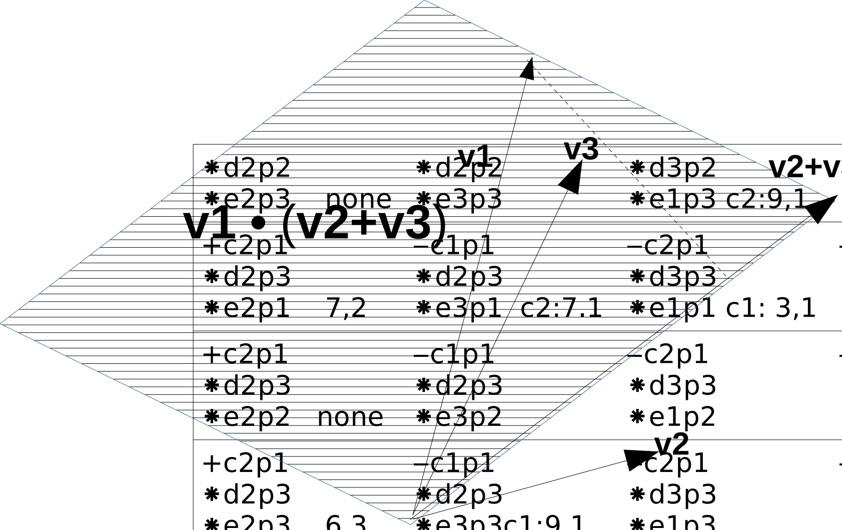


c2px✱d2px = c3px✱d1px
c2px✱e2px = c1px✱e3px
d2px✱e2px = d3px✱e1px

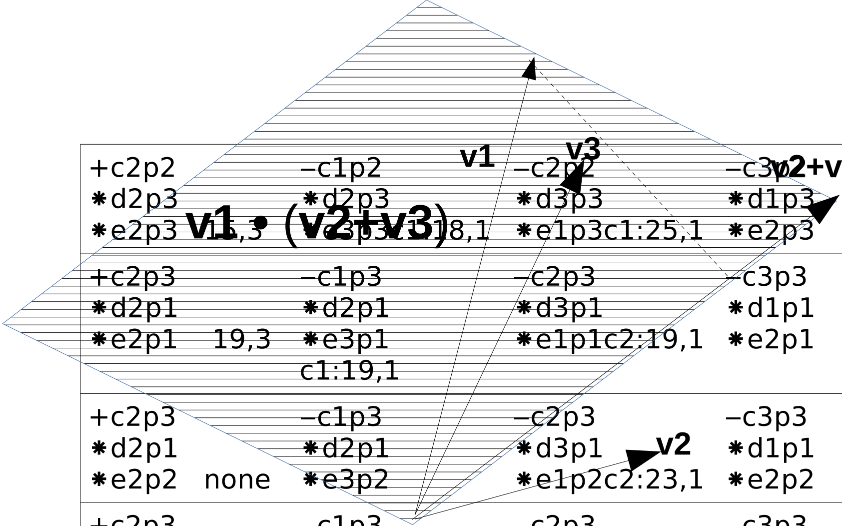
c2p1 = q111✱q221
d2p1 = q121✱q231
e2p1 = q131✱q211
c3p1 = q111✱q231
d1p1 = q121✱q221
c1p1 = q111✱q211
e3p1 = q131✱q221
d3p1 = q121✱q211
e1p1 = q131✱q231
c2p2 = q112✱q222
d2p2 = q122✱q232

c3p2 = q112✱q232
d1p2 = q122✱q222

+c2p1 ✱d2p1 ✱e2p1 1,3	−c1p1 ✱d2p1 ✱e3p1 c1: 1,1	−c2p1 ✱d3p1 ✱e1p1 c2: 1,1	−c3p1 ✱d1p1 ✱e2p1
+c2p1 ✱d2p1 ✱e2p2 2,4	−c1p1 ✱d2p1 ✱e3p2	−c2p1 ✱d3p1 ✱e1p2	−c3p1 ✱d1p1 ✱e2p2 c2:2,1
+c2p1 ✱d2p1 ✱e2p3 3,4	−c1p1 ✱d2p1 ✱e3p3	−c2p1 ✱d3p1 ✱e1p2	−c3p1 ✱d1p1 ✱e2p3 c2:3,1
+c2p1 ✱d2p2 ✱e2p1 4,2	−c1p1 ✱d2p2 ✱e3p1 c2:4,1	−c2p1 ✱d3p2hh ✱e1p1 c1:2,1	−c3p1 ✱d1p2 ✱e2p1 c1:4,1
+c2p1 ✱d2p2 ✱e2p2 5,3	−c1p1 ✱d2p2 ✱e3p2 c1:5,1	−c2p1 ✱d3p2 ✱e1p2 c2:5,1	−c3p1 ✱d1p2 ✱e2p2
+c2p1	−c1p1	−c2p1	−c3p1

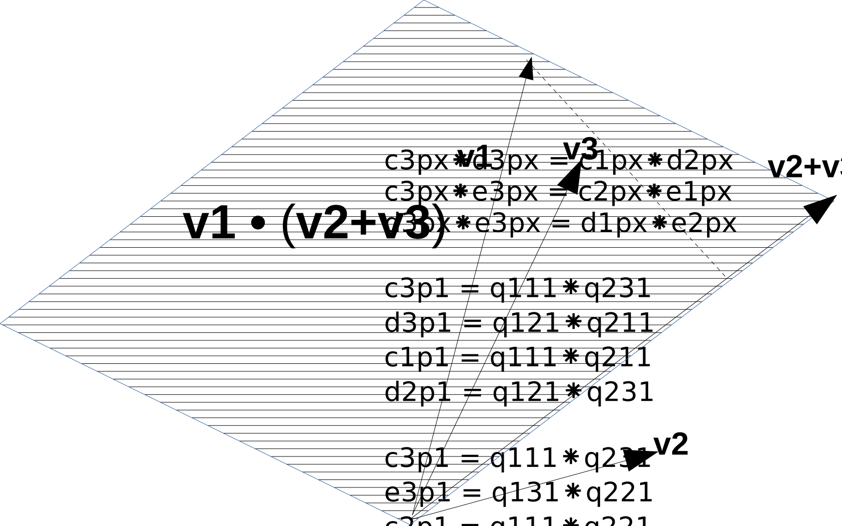


✱d2p2 ✱e2p3 none	✱d2p2 ✱e3p3	✱d3p2 ✱e1p3 c2:9,1	✱d1p2 ✱e2p3
+c2p1 ✱d2p3 ✱e2p1 7,2	−c1p1 ✱d2p3 ✱e3p1 c2:7.1	−c2p1 ✱d3p3 ✱e1p1 c1: 3,1	−c3p1 ✱d1p3 ✱e2p1 c1:7,1
+c2p1 ✱d2p3 ✱e2p2 none	−c1p1 ✱d2p3 ✱e3p2	−c2p1 ✱d3p3 ✱e1p2	−c3p1 ✱d1p3 ✱e2p2
+c2p1 ✱d2p3 ✱e2p3 6,3	−c1p1 ✱d2p3 ✱e3p3c1:9,1	−c2p1 ✱d3p3 ✱e1p3	−c3p1 ✱d1p3 ✱e2p3
+c2p2 ✱d2p1 ✱e2p1 10,3	−c1p2 ✱d2p1 ✱e3p1 c1:10,1	−c2p2 ✱d3p1 ✱e1p1c2:10,1	−c3p2 ✱d1p1 ✱e2p1
+c2p2 ✱d2p1 ✱e2p2 11,2	−c1p2 ✱d2p1 ✱e3p2c2:11,1	−c2p2 ✱d3p1 ✱e1p2c1:13,1	−c3p2 ✱d1p1 ✱e2p2 c1:11,1
+c2p2 ✱d2p1 ✱e2p3 none	−c1p2 ✱d2p1 ✱e3p3	−c2p2 ✱d3p1 ✱e1p3	−c3p2 ✱d1p1 ✱e2p3
+c2p2 ✱d2p2 ✱e2p1 13,4	−c1p2 ✱d2p2 ✱e3p1	−c2p2 ✱d3p2 ✱e1p1	−c3p2 ✱d1p2 ✱e2p1c2:13,1
+c2p2 ✱d2p2 ✱e2p2 14,3	−c1p2 ✱d2p2 ✱e3p2 c1:14,1	−c2p2 ✱d3p2 ✱e1p2c2:14,1	−c3p2 ✱d1p2 ✱e2p2
+c2p2 ✱d2p2 ✱e2p3 15,4	−c1p2 ✱d2p2 ✱e3p3	−c2p2 ✱d3p2 ✱e1p3c2:18,1	−c3p2 ✱d1p2 ✱e2p3c2:15,1
+c2p2 ✱d2p3 ✱e2p1 none4	−c1p2 ✱d2p3 ✱e3p1	−c2p2 ✱d3p3 ✱e1p1	−c3p2 ✱d1p3 ✱e2p1
+c2p2 ✱d2p3 ✱e2p2 17,2	−c1p2 ✱d2p3 ✱e3p2c2:17,1	−c2p2 ✱d3p3 ✱e1p2c1:15,1	−c3p2 ✱d1p3 ✱e2p2c1:17,1



+c2p2 ✱d2p3 ✱e2p3 18,3	−c1p2 ✱d2p3 ✱e3p3c1:18,1	−c2p2 ✱d3p3 ✱e1p3c1:25,1	−c3p2 ✱d1p3 ✱e2p3
+c2p3 ✱d2p1 ✱e2p1 19,3	−c1p3 ✱d2p1 ✱e3p1 c1:19,1	−c2p3 ✱d3p1 ✱e1p1c2:19,1	−c3p3 ✱d1p1 ✱e2p1
+c2p3 ✱d2p1 ✱e2p2 none	−c1p3 ✱d2p1 ✱e3p2	−c2p3 ✱d3p1 ✱e1p2c2:23,1	−c3p3 ✱d1p1 ✱e2p2
+c2p3 ✱d2p1 ✱e2p3 21,2	−c1p3 ✱d2p1 ✱e3p3c2:21,1	−c2p3 ✱d3p1 ✱e1p3c1:26,1	−c3p3 ✱d1p1 ✱e2p3c1:21,1
+c2p3 ✱d2p2 ✱e2p1 none	−c1p3 ✱d2p2 ✱e3p1	−c2p3 ✱d3p2 ✱e1p1	−c3p3 ✱d1p2 ✱e2p1
+c2p3 ✱d2p2 ✱e2p2 20,3	−c1p3 ✱d2p2 ✱e3p2c1:23,1	−c2p3 ✱d3p2 ✱e1p2	−c3p3 ✱d1p2 ✱e2p2
+c2p3 ✱d2p2 ✱e2p3 24,2	−c1p3 ✱d2p2 ✱e3p3c2:24,1	−c2p3 ✱d3p2 ✱e1p3	−c3p3 ✱d1p2 ✱e2p3c1:24,1
+c2p3 ✱d2p3 ✱e2p1 25,4	−c1p3 ✱d2p3 ✱e3p1	−c2p3 ✱d3p3 ✱e1p1	−c3p3 ✱d1p3 ✱e2p1c2:25,1
+c2p3 ✱d2p3 ✱e2p2 26,4	−c1p3 ✱d2p3 ✱e3p2	−c2p3 ✱d3p3 ✱e1p2	−c3p3 ✱d1p3 ✱e2p2c2:26,1
+c2p3 ✱d2p3 ✱e2p3 27,3	−c1p3 ✱d2p3 ✱e3p3c1:27,1	−c2p3 ✱d3p3 ✱e1p3c2:27,1	−c3p3 ✱d1p3 ✱e2p3
21	6	9	6

Result: Out of 27 positive c2's
21 are canceled by negatives
6 c1's 9 c2's 6 c3's
6 are left unmatched.



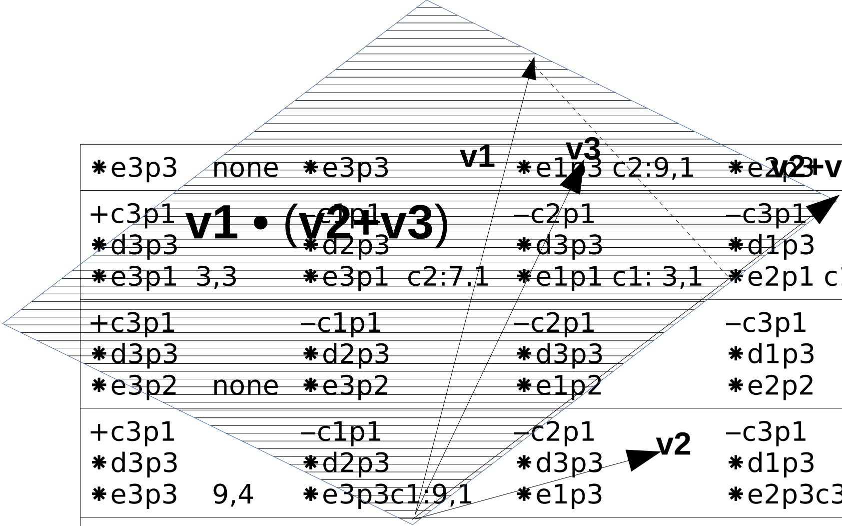
c3px✱d3px = c1px✱d2px
c3px✱e3px = c2px✱e1px
d3px✱e3px = d1px✱e2px

c3p1 = q111✱q231
d3p1 = q121✱q211
c1p1 = q111✱q211
d2p1 = q121✱q231

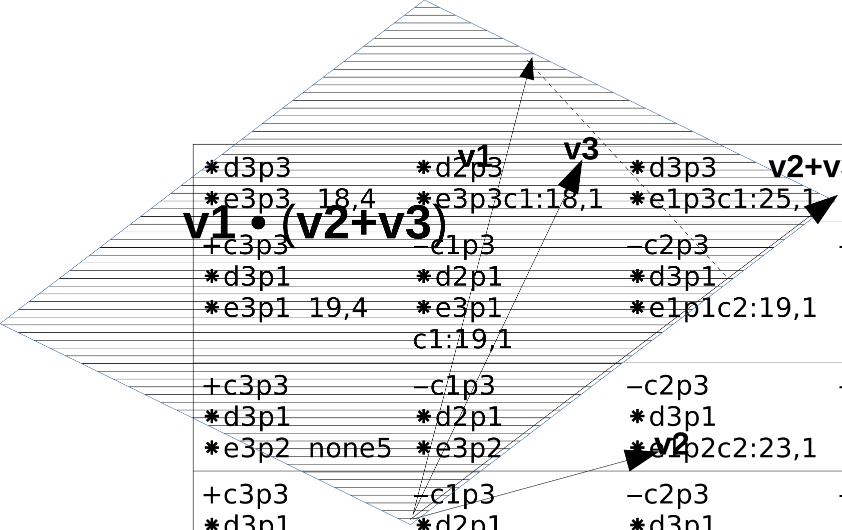
c3p1 = q111✱q231
e3p1 = q131✱q221
c2p1 = q111✱q221
e1p1 = q131✱q231

d3p1 = q121✱q211
e3p1 = q131✱q221
d1p1 = q121✱q221
e2p1 = q131✱q211

+c3p1 ✱d3p1 ✱e3p1 1,4	−c1p1 ✱d2p1 ✱e3p1 c1: 1,1	−c2p1 ✱d3p1 ✱e1p1 c2: 1,1	−c3p1 ✱d1p1 ✱e2p1c3: 1,1
+c3p1 ✱d3p1 ✱e3p2 2,2	−c1p1 ✱d2p1 ✱e3p2 c3:2,1	−c2p1 ✱d3p1 ✱e1p2	−c3p1 ✱d1p1 ✱e2p2 c2:2,1
+c3p1 ✱d3p1 ✱e3p3 3,2	−c1p1 ✱d2p1 ✱e3p3c3:3,1	−c2p1 ✱d3p1 ✱e1p2	−c3p1 ✱d1p1 ✱e2p3 c2:3,1
+c3p1 ✱d3p2 ✱e3p1 2,3	−c1p1 ✱d2p2 ✱e3p1 c2:4,1	−c2p1 ✱d3p2hh ✱e1p1 c1:2,1	−c3p1 ✱d1p2 ✱e2p1 c1:4,1
+c3p1 ✱d3p2 ✱e3p2 5,4	−c1p1 ✱d2p2 ✱e3p2 c1:5,1	−c2p1 ✱d3p2 ✱e1p2 c2:5,1	−c3p1 ✱d1p2 ✱e2p2c3:5,1
+c3p1 ✱d3p2	−c1p1 ✱d2p2	−c2p1 ✱d3p2	−c3p1 ✱d1p2

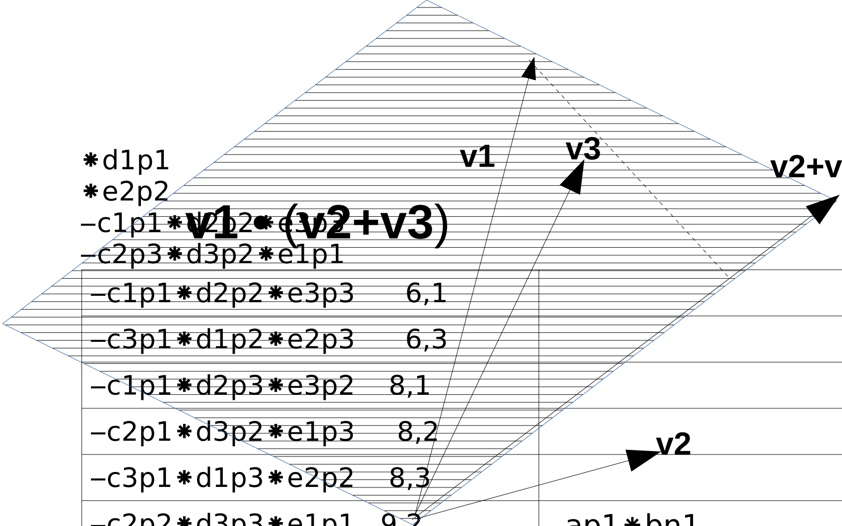


✱e3p3 none	✱e3p3	✱e1p3 c2:9,1	✱e2p3
+c3p1 ✱d3p3 ✱e3p1 3,3	−c1p1 ✱d2p3 ✱e3p1 c2:7.1	−c2p1 ✱d3p3 ✱e1p1 c1: 3,1	−c3p1 ✱d1p3 ✱e2p1 c1:7,1
+c3p1 ✱d3p3 ✱e3p2 none	−c1p1 ✱d2p3 ✱e3p2	−c2p1 ✱d3p3 ✱e1p2	−c3p1 ✱d1p3 ✱e2p2
+c3p1 ✱d3p3 ✱e3p3 9,4	−c1p1 ✱d2p3 ✱e3p3c1:9,1	−c2p1 ✱d3p3 ✱e1p3	−c3p1 ✱d1p3 ✱e2p3c3:9,1
10+c3p2 ✱d3p1 ✱e3p1 10,4	−c1p2 ✱d2p1 ✱e3p1 c1:10,1	−c2p2 ✱d3p1 ✱e1p1c2:10,1	−c3p2 ✱d1p1 ✱e2p1c3:10,1
+c3p2 ✱d3p1 ✱e3p2 13,3	−c1p2 ✱d2p1 ✱e3p2c2:11,1	−c2p2 ✱d3p1 ✱e1p2c1:13,1	−c3p2 ✱d1p1 ✱e2p2 c1:11,1
+c3p2 ✱d3p1 ✱e3p3 none	−c1p2 ✱d2p1 ✱e3p3	−c2p2 ✱d3p1 ✱e1p3	−c3p2 ✱d1p1 ✱e2p3
+c3p2 ✱d3p2 ✱e3p1 13,2	−c1p2 ✱d2p2 ✱e3p1c3:13,1	−c2p2 ✱d3p2 ✱e1p1	−c3p2 ✱d1p2 ✱e2p1c2:13,1
+c3p2 ✱d3p2 ✱e3p2 14,4	−c1p2 ✱d2p2 ✱e3p2 c1:14,1	−c2p2 ✱d3p2 ✱e1p2c2:14,1	−c3p2 ✱d1p2 ✱e2p2c3:14,1
+c3p2 ✱d3p2 ✱e3p3 15,2	−c1p2 ✱d2p2 ✱e3p3c3:15,1	−c2p2 ✱d3p2 ✱e1p3c2:18,1	−c3p2 ✱d1p2 ✱e2p3c2:15,1
+c3p2 ✱d3p3 ✱e3p1 none4	−c1p2 ✱d2p3 ✱e3p1	−c2p2 ✱d3p3 ✱e1p1	−c3p2 ✱d1p3 ✱e2p1
+c3p2 ✱d3p3 ✱e3p2 12,3	−c1p2 ✱d2p3 ✱e3p2c2:17,1	−c2p2 ✱d3p3 ✱e1p2c1:15,1	−c3p2 ✱d1p3 ✱e2p2c1:17,1
+c3p2	−c1p2	−c2p2	−c3p2



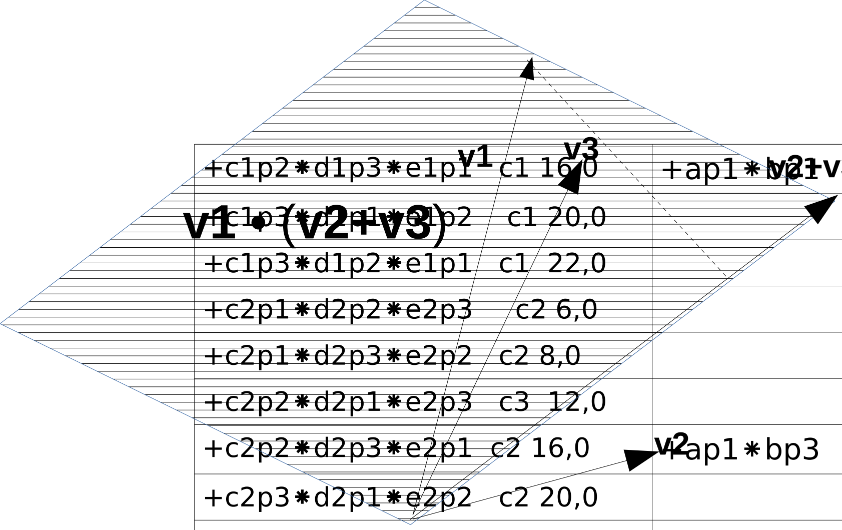
✱d3p3 ✱e3p3 18,4	✱d2p3 ✱e3p3c1:18,1	✱d3p3 ✱e1p3c1:25,1	✱d1p3 ✱e2p3c3:18,1
+c3p3 ✱d3p1 ✱e3p1 19,4	−c1p3 ✱d2p1 ✱e3p1 c1:19,1	−c2p3 ✱d3p1 ✱e1p1c2:19,1	−c3p3 ✱d1p1 ✱e2p1c3:19,1
+c3p3 ✱d3p1 ✱e3p2 none5	−c1p3 ✱d2p1 ✱e3p2	−c2p3 ✱d3p1 ✱e1p2c2:23,1	−c3p3 ✱d1p1 ✱e2p2
+c3p3 ✱d3p1 ✱e3p3 25,3	−c1p3 ✱d2p1 ✱e3p3c2:21,1	−c2p3 ✱d3p1 ✱e1p3c1:26,1	−c3p3 ✱d1p1 ✱e2p3c1:21,1
+c3p3 ✱d3p2 ✱e3p1 none6	−c1p3 ✱d2p2 ✱e3p1	−c2p3 ✱d3p2 ✱e1p1	−c3p3 ✱d1p2 ✱e2p1
+c3p3 ✱d3p2 ✱e3p2 23,4	−c1p3 ✱d2p2 ✱e3p2c1:23,1	−c2p3 ✱d3p2 ✱e1p2	−c3p3 ✱d1p2 ✱e2p2 c3:23,1
+c3p3 ✱d3p2 ✱e3p3 23,3	−c1p3 ✱d2p2 ✱e3p3c2:24,1	−c2p3 ✱d3p2 ✱e1p3	−c3p3 ✱d1p2 ✱e2p3c1:24,1
+c3p3 ✱d3p3 ✱e3p1 25,2	−c1p3 ✱d2p3 ✱e3p1 c3:25,1	−c2p3 ✱d3p3 ✱e1p1	−c3p3 ✱d1p3 ✱e2p1c2:25,1
+c3p3 ✱d3p3 ✱e3p2 26,2	−c1p3 ✱d2p3 ✱e3p2 c3:26,1	−c2p3 ✱d3p3 ✱e1p2	−c3p3 ✱d1p3 ✱e2p2c2:26,1
+c3p3 ✱d3p3 ✱e3p3 27,4	−c1p3 ✱d2p3 ✱e3p3c1:27,1	−c2p3 ✱d3p3 ✱e1p3c2:27,1	−c3p3 ✱d1p3 ✱e2p3 c3:27,1
21	6	6	9

Result: Out of 27 positive c3's
21 are canceled by negatives
6 c1's 9 c2's 6 c3's
6 are left unmatched.
−c3p3



✱d1p1
✱e2p2
−c1p1✱d2p2✱e3p2
−c2p3✱d3p2✱e1p1

−c1p1✱d2p2✱e3p3 6,1	
−c3p1✱d1p2✱e2p3 6,3	
−c1p1✱d2p3✱e3p2 8,1	
−c2p1✱d3p2✱e1p3 8,2	
−c3p1✱d1p3✱e2p2 8,3	
−c2p2✱d3p3✱e1p1 9,2	−ap1✱bn1
−c1p2✱d2p1✱e3p3 12,1	
−c3p2✱d1p1✱e2p3 12,3	
−c1p2✱d2p3✱e3p1 16,1	–ap1✱bn3
−c2p2✱d3p1✱e1p3 16,2	
−c3p2✱d1p3✱e2p1 16,3	–ap1✱bn2
−c1p3✱d2p1✱e3p2 20,1	
−c3p3✱d1p1✱e2p2 20,3	
−c1p3✱d2p2✱e3p1 22,1	
−c2p3✱d3p1✱e1p2 22,2	
−c3p3✱d1p2✱e2p1 22,3	
−c2p3✱d3p3✱e1p3 24,2	hh
−c2p3✱d3p2✱e1p3 26,2	hh
+c3p1✱d3p2✱e3p3 c3 6,0	
+c3p1✱d3p3✱e3p2 8,0	
+c3p2✱d3p1✱e3p3 12,0	
+c3p2✱d3p3✱e3p1 16,0	+ap1✱bp2
+c3p3✱d3p1✱e3p2 20,0	
+c3p3✱d3p2✱e3p1 22,0	
+c1p1✱d1p2✱e1p3 c1 6,0	
+c1p1✱d1p3✱e1p2 c1 8,0	
+c1p2✱d1p1✱e1p3 c1 12,0	



+c1p2∗d1p3∗e1p1 c1 16,0	+ap1∗bp1
+c1p3∗d1p1∗e1p2 c1 20,0	
+c1p3∗d1p2∗e1p1 c1 22,0	
+c2p1∗d2p2∗e2p3 c2 6,0	
+c2p1∗d2p3∗e2p2 c2 8,0	
+c2p2∗d2p1∗e2p3 c3 12,0	
+c2p2∗d2p3∗e2p1 c2 16,0	+ap1∗bp3
+c2p3∗d2p1∗e2p2 c2 20,0	
+c2p3∗d2p2∗e2p1 c2 22, 0	

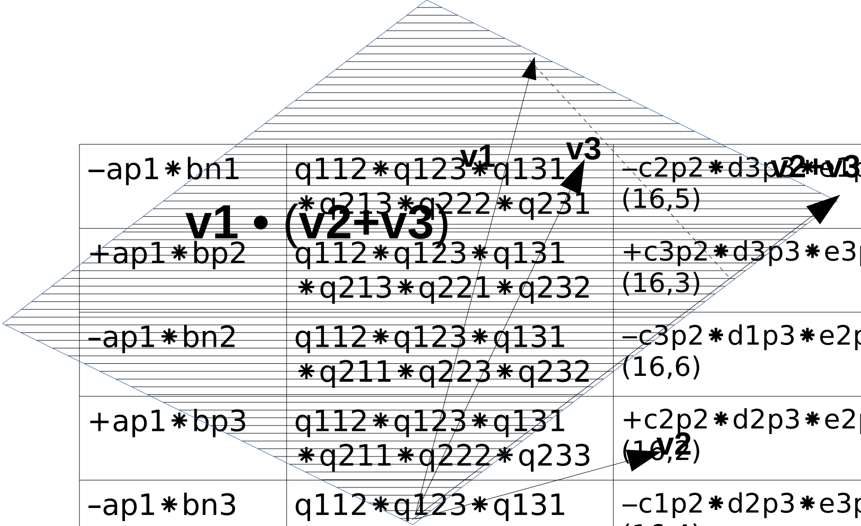
6+
18-

+ap1∗bp3+c2p2∗d2p3∗e2p1

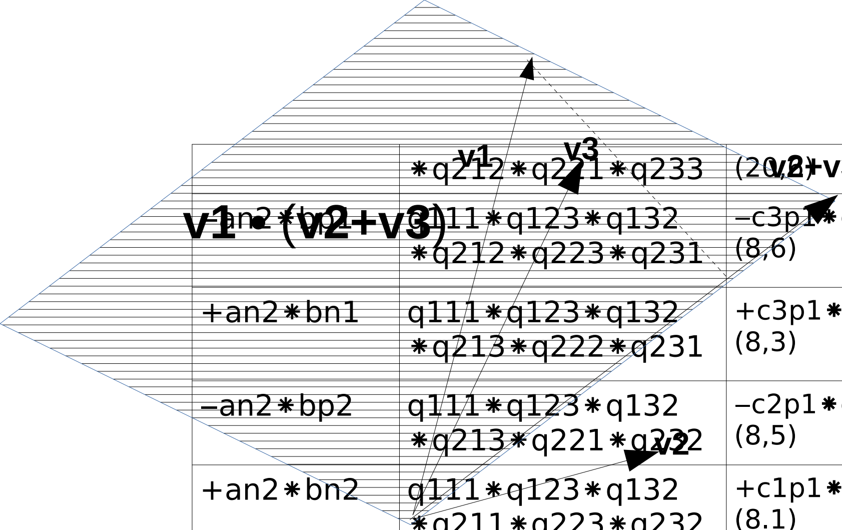
c1p3 = q113∗q213
c2p3 = q113∗q223
c3p3 = q113∗q233
d1p2 = q122∗q222
d2p2 = q122∗q232
d2p3 = q123∗q233
e1p1 = q131∗q231
e1p2 = q132∗q232
e1p3 = q133∗q233
e2p1 = q131∗q211
e2p2 = q132∗q212
e2p3 = q133∗q213
e3p1 = q131∗q221

c1p2 = q112∗q212
d1p3 = q123∗q223

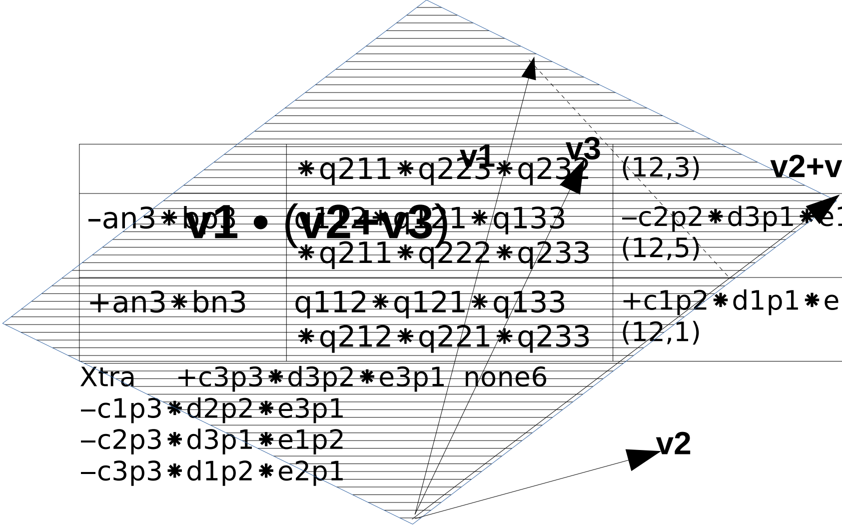
headings		
+ap1∗bp1	q112∗q123∗q131 ∗q212∗q223∗q231	+c1p2∗d1p3∗e1p1 (16,1)



−ap1∗bn1	q112∗q123∗q131 ∗q213∗q222∗q231	−c2p2∗d3p3∗e1p1 (16,5)
+ap1∗bp2	q112∗q123∗q131 ∗q213∗q221∗q232	+c3p2∗d3p3∗e3p1 (16,3)
−ap1∗bn2	q112∗q123∗q131 ∗q211∗q223∗q232	−c3p2∗d1p3∗e2p1 (16,6)
+ap1∗bp3	q112∗q123∗q131 ∗q211∗q222∗q233	+c2p2∗d2p3∗e2p1 (16,2)
−ap1∗bn3	q112∗q123∗q131 ∗q212∗q221∗q233	−c1p2∗d2p3∗e3p1 (16,4)
−an1∗bp1	q113∗q122∗q131 ∗q212∗q223∗q231	−c2p3∗d3p2∗e1p1 (22,5)
+an1∗bn1	q113∗q122∗q131 ∗q213∗q222∗q231	+c1p3∗d1p2∗e1p1 (22,1)
−an1∗bp2	q113∗q122∗q131 ∗q213∗q221∗q232	−c1p3∗d2p2∗e3p1 (22,4)
+an1∗bn2	q113∗q122∗q131 ∗q211∗q223∗q232	+c2p3∗d2p2∗e2p1 (22,2)
−an1∗bp3	q113∗q122∗q131 ∗q211∗q222∗q233	−c3p3∗d1p2∗e2p1 (22,6)
+an1∗bn3	q113∗q122∗q131 ∗q212∗q221∗q233	+c3p3∗d3p2∗e3p1 (22,3)
+ap2∗bp1	q113∗q121∗q132 ∗q212∗q223∗q231	+c2p3∗d2p1∗e2p2 (20,2)
−ap2∗bn1	q113∗q121∗q132 ∗q213∗q222∗q231	−c1p3∗d2p1∗e3p2 (20,4)
+ap2∗bp2	q113∗q121∗q132 ∗q213∗q221∗q232	+c1p3∗d1p1∗e1p2 (20,1)
−ap2∗bn2	q113∗q121∗q132 ∗q211∗q223∗q232	−c2p3∗d3p1∗e1p2 (20,5)
+ap2∗bp3	q113∗q121∗q132 ∗q211∗q222∗q233	+c3p3∗d3p1∗e3p2 (20,3)
−ap2∗bn3	q113∗q121∗q132	−c3p3∗d1p1∗e2p2



	∗q212∗q221∗q233	(20,2)
−an2∗bp1	q111∗q123∗q132 ∗q212∗q223∗q231	−c3p1∗d1p3∗e2p2 (8,6)
+an2∗bn1	q111∗q123∗q132 ∗q213∗q222∗q231	+c3p1∗d3p3∗e3p2 (8,3)
−an2∗bp2	q111∗q123∗q132 ∗q213∗q221∗q232	−c2p1∗d3p3∗e1p2 (8,5)
+an2∗bn2	q111∗q123∗q132 ∗q211∗q223∗q232	+c1p1∗d1p3∗e1p2 (8.1)
−an2∗bp3	q111∗q123∗q132 ∗q211∗q222∗q233	−c1p1∗d2p3∗e3p2 (8,4)
+an2∗bn3	q111∗q123∗q132 ∗q212∗q221∗q233	+c2p1∗d2p3∗e2p2 (8,2)
+ap3∗bp1	q111∗q122∗q133 ∗q212∗q223∗q231	+c3p1∗d3p2∗e3p3 (6,3)
–ap3∗bn1	q111∗q122∗q133 ∗q213∗q222∗q231	−c3p1∗d1p2∗e2p3 (6,6)
+ap3∗bp2	q111∗q122∗q133 ∗q213∗q221∗q232	+c2p1∗d2p2∗e2p3 (6,2)
–ap3∗bn2	q111∗q122∗q133 ∗q211∗q223∗q232	−c1p1∗d2p2∗e3p3 (6,4)
+ap3∗bp3	q111∗q122∗q133 ∗q211∗q222∗q233	+c1p1∗d1p2∗e1p3 (6,1)
–ap3∗bn3	q111∗q122∗q133 ∗q212∗q221∗q233	+c2p1∗d3p2∗e1p3 (6,5)
−an3∗bp1	q112∗q121∗q133 ∗q212∗q223∗q231	−c1p2∗d2p1∗e3p3 (12,4)
+an3∗bn1	q112∗q121∗q133 ∗q213∗q222∗q231	+c2p2∗d2p1∗e2p3 (12,2)
–an3∗bp2	q112∗q121∗q133 ∗q213∗q221∗q232	+c3p2∗d1p1∗e2p3 (12,6)
+an3∗bn2	q112∗q121∗q133	+c3p2∗d3p1∗e3p3



	∗q211∗q223∗q232	(12,3)
−an3∗bn3	q112∗q121∗q133 ∗q211∗q222∗q233	−c2p2∗d3p1∗e1p3 (12,5)
+an3∗bn3	q112∗q121∗q133 ∗q212∗q221∗q233	+c1p2∗d1p1∗e1p3 (12,1)

Xtra +c3p3∗d3p2∗e3p1 none6
−c1p3∗d2p2∗e3p1
−c2p3∗d3p1∗e1p2
−c3p3∗d1p2∗e2p1

Use only character sets rather than different font styles Only font styes used will be Tahoma and Garamond ITC T

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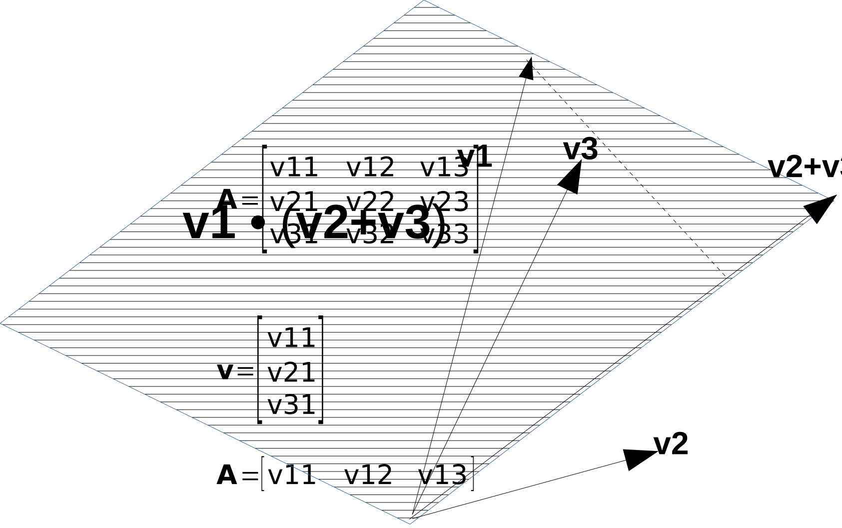
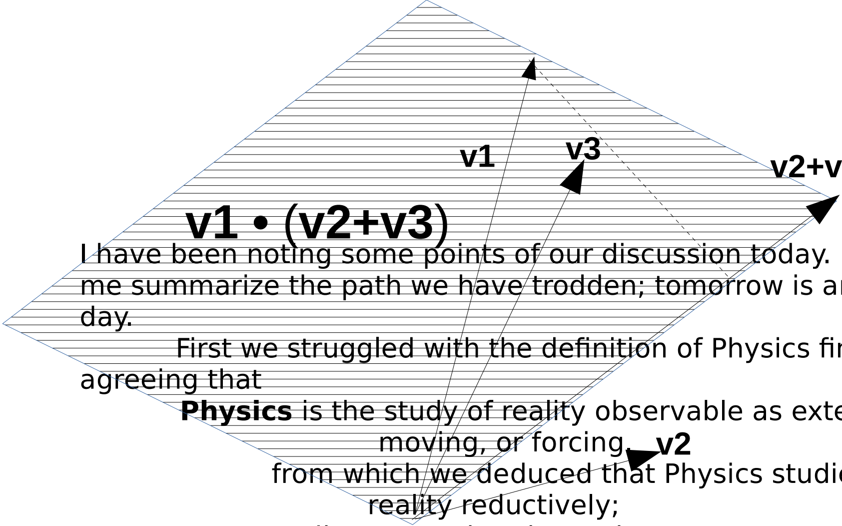

v1
v3
v2+v3
A = [v11 v12 v13; v21 v22 v23; v31 v32 v33]
v1 • (v2+v3)
v = [v11; v21; v31]
v2
A = [v11 v12 v13]



v1 • (v2+v3)

I have been noting some points of our discussion today. Let me summarize the path we have trodden; tomorrow is another day.

First we struggled with the definition of Physics finally agreeing that

Physics is the study of reality observable as extended,
moving, or forcing,
from which we deduced that Physics studies
reality reductively;
reality comprehends much more.

We also proved God exists and so becomes
a context of all physical inquiry.

Then we noted the confusion
between mathematical symbols
and similar symbols used for Physics.

We discussed why mathematics as such is
inappropriate for Physics.

So we started a discussion of theoretical physics
with a discussion on symbols.

Although symbols themselves are arbitrary,
they must follow some rules to avoid
becoming ambiguous or misleading.

We settled on a set of rules for the symbols.

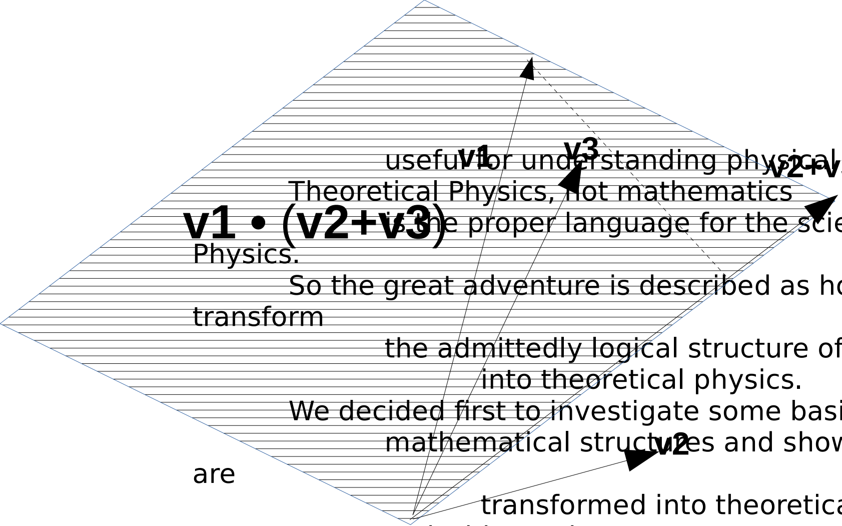
Then Einstein asked a question about variables
which led to a discussion on how to avoid
typical ambiguities.

We decided to use the following symbols
x for the object being observed
r for the position of the object
a to denote the observation.

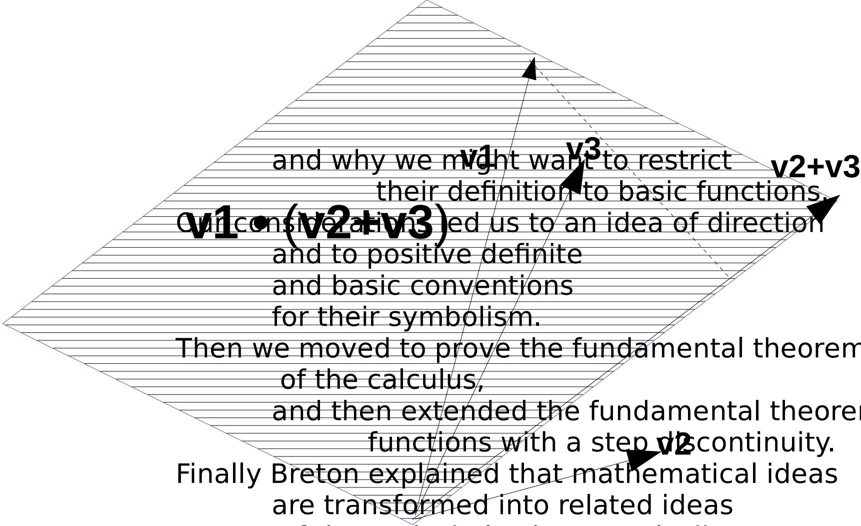
We then discussed the difference between
equations and definitions. We agreed to use
≡ for definitions
and
= for equations

Then we saw that words 'set' and 'operator'
have different mathematical and physical meanings.

Then we defined theoretical physics as
a set of coherent ideas related
in a framework of logically consistent
statements



useful for understanding physical observations.
Theoretical Physics, not mathematics
is the proper language for the science of
Physics.
So the great adventure is described as how to
transform
the admittedly logical structure of mathematics
into theoretical physics.
We decided first to investigate some basic
mathematical structures and show how these
are
transformed into theoretical physics.
We started with numbers.
In so common an idea as the positive integers
we discovered oceans of fascination
which continued to negative integers
and then to quotient numbers.
We also saw ways in which the mathematical ideas
differ
from those of theoretical physics,
even when symbolized identically.
Then we launched into a discussion of functions.
In the set of all functions,
those most interesting to theoretical physics
form
only a minuscule part.
For these restricted functions we looked at limits
and those which map topologies into topologies
from which we learned about
continuous functions
and look−alike functions.
With the restrictions also came qualities which could
prove useful for the study for Physics.
We defined other categories of functions:
step,
linear,
summed
multiplied,
restricted,
compound.
We then took up the special limiting
functions of functions namely
differentiation
and integration,



and why we might want to restrict
their definition to basic functions.
Our considerations led us to an idea of direction
and to positive definite
and basic conventions
for their symbolism.
Then we moved to prove the fundamental theorem
of the calculus,
and then extended the fundamental theorem to
functions with a step discontinuity.
Finally Breton explained that mathematical ideas
are transformed into related ideas
of theoretical physics generically
by first restricting the idea
which, upon restriction, may then be
expanded to a large panoply of related ideas.
It is these ideas, not the mathematical ones,
which are suitable for Physics.
He showed how this applies to physical variables,
and the rules governing physical units.
The primary physical variables for theoretical physics are
extension, motion, and movers,
which are idealizes as length, velocity, and force.
References are required for theoretical physics,
but not for mathematics.
Material and spatial references need to be distinguished.
Breton promised our adventure would look to finding
relationships between the two.
We discussed how material sets differ
from mathematical sets
and how the topological ideas of mathematics
are transformed into theoretical physics.
In particular the idea of a particle in theoretical physics
differs from a mathematical point.
We showed how physical properties can be attached to
particles as ideas,
and how the idea of a particle accommodates
the physical process of observation
with limited resolution."